Modular-invariance in rational conformal field theory: past, present and future: Lecture 2

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Overview

- representations of VOAs
- rational VOAs
- examples of rational VOAs
- the modular-invariance conjecture
- C_2-cofinite and regular VOAs
- the associated vector-valued modular form
- holomorphic VOAs
- approach using MTCs
- approach using vector-valued modular forms
- unbounded denominators and ASD conjecture

Representations of VOAs

Representation theory of algebraic objects is usually complicated and technical. This is especially true of VOAs.

The good news is that there *is* a natural, deep and rich representation theory. It is the origin of all questions to do with modularity in VOA theory.

The not-so-good news is that it is treacherous. In part, this is because of the many different kinds of V-modules. We have the usual types:

simple, completely reducible, indecomposable

projective, injective, ...

In VOA theory, modules usually have some kind of grading. This is an important feature of V-modules. But there are many types of gradings that can occur: gradings by $\mathbb{Z}$ (as for a VOA itself), and more general subsets of $\mathbb{C}$. The graded pieces may or may not be finite-dimensional, and the grading may or may not be bounded below (if that makes sense).

Moreover, one needs to distinguish between modules over a VA versus modules over a VOA.

All of this makes it impossible to give an adequate survey of representation theory here.

We give just a few of the basic ideas, with little precision.

V-mod

Objects $A \in \mathbf{V} - \mathbf{mod}$ are linear spaces with a *state-field correspondence* of mutually local, translation-covariant fields

$$Y = Y_A : V \longrightarrow \mathfrak{F}(A)$$
$$u \mapsto Y_A(u,z) = \sum_n u_A(n) z^{-n-1}$$
$$Y_A(u,z) \sim Y_A(v,z)$$

and a grading into $L_A(0)$-eigenspaces

$$A = \bigoplus_{\lambda \in \mathbb{C}} A_\lambda$$

If A is a *simple* object, its grading takes the form

$$A = \bigoplus_{n \geq 0} A_{h+n} \qquad (\dim A_{h+n} < \infty)$$

for some $h \in \mathbb{C}$ called the *conformal weight* of A.

Rational VOAs

The next result shows how different the representation theory of VOAs can be compared to that of Lie algebras.

Theorem If $\mathbf{V} - \mathbf{mod}$ is *semisimple*, i.e., 'admissible' V-modules (not quite defined here) are direct sums of simple objects, then the set of iso classes of simple objects is *finite*.

VOAs such that $\mathbf{V} - \mathbf{mod}$ is semisimple are called *rational*.

We have the partition function of a simple module

$$Z_A(q) := q^{h-c/24} \sum_{n \geq 0} dim A_{h+n} q^n$$

Let's look at the previous 4 examples of VOAs.

Heisenberg modules

Theorem The Heisenberg VOA $M(1)$ is not rational. It has a *unique* simple module for each conformal weight h.

This is a VOA version of the Stone-von Neumann theorem.

Virasoro modules

Theorem The Virasoro VOA V_c of central charge c is *never* rational.

V_c has a maximal ideal $J_c \subseteq V_c$ and

$$\begin{aligned} J_c \neq 0 \Leftrightarrow V_c/J_c \quad &\text{is rational} \\ \Leftrightarrow c = 1 - &\frac{6(p-q)^2}{pq} \\ (gcd(p,q) = 1, 2 \leq &p < q) \end{aligned}$$

$J_c \neq 0 \Rightarrow$ there are exactly $\frac{1}{2}(p-1)(q-1)$ iso classes of simple modules A for V_c/J_c. In each case

$Z_A(q)$ is a modular function of weight 0

on a congruence subgroup

Moonshine modules

Theorem The Moonshine module $V^\natural$ is simple. Moreover, this is the *unique* simple $V^\natural$-module.

The partition function is the modular invariant

$$\begin{aligned} Z_{V^\natural}(q) &= j(q) - 744 \\ &= q^{-1} + 196884q + \ldots \end{aligned}$$

Lattice theory modules

Theorem L is an even lattice of rank ℓ. Every lattice theory V_L is rational. The simple modules are indexed by the cosets L^*/L of L in its *dual lattice*.

If $L + \gamma \in L^*/L$ then

$$Z_{L+\gamma}(q) = \frac{\theta_{L+\gamma}(q)}{\eta^{\ell}}$$

is a modular function of weight 0 on a congruence subgroup.

The Mother of all modular-invariance conjectures

Let V be a rational VOA. It therefore has only *finitely many* iso classes of simple modules, with representatives $A^1, \ldots, A^d$ and conformal weights $h_1, \ldots, h_d$. Recall that this means that the $L(0)$-grading on A^j is

$$A^j = \oplus_{n \geq 0} A_{h_j+n}$$

Here is the basic conjecture.

$Z_{A^j}(q) := q^{h_j - c/24} \sum_n dim A_{h_j+n} q^n$ is a modular function of weight 0 on a congruence subgroup

At this level of generality, this is wide open!

C_2-cofinite VOAs

Most progress on the modular-invariance conjecture has been made assuming additional conditions.

The VOA V is called *C_2-cofinite* if

$$codim \ \langle u(-2)v | u, v \in V \rangle < \infty$$

Here is another basic conjecture.

$$\boxed{V \text{ rational} \Rightarrow V \ C_2\text{-cofinite}}$$

'Theorem' If V is C_2-cofinite, the partition function $Z_A(q)$ for a simple module A is *holomorphic for* $\tau \in \mathcal{H}$ if we set $q = e^{2\pi i \tau}$.

This can be proved either by directly estimating the growth of $dimA_{h+n}$, or by showing that $Z_A(q)$ satisfies a differential equation with coefficient functions holomorphic in $\mathcal{H}$ and a regular-singularity at $q = 0$. Then Frobenius-Fuchs theory applies. This technique will be important in several places below.

(Actually, only somewhat weaker forms of the last Theorem are in the literature. This is what 'Theorem' means. However, I believe the 'Theorem' as stated can be proved!)

Regular VOAs

Call V *regular* if it is *both* rational and C_2-cofinite. (According to the last conjecture, this is the *same* as rationality.)

For a regular VOA we therefore have finitely many simple modules, with *convergent* partition functions

$$Z_{A^j}(\tau) = q^{h_j - c/24} \sum_n dim A_{h+n} q^n \quad (\tau \in \mathcal{H})$$

Theorem Let V be a regular VOA. The central charge c and conformal weights h_j lie in $\mathbb{Q}$.

The idea is again to use Frobenius-Fuchs theory. We say more about this shortly.

Let $N > 0$ be the common denominator of the rationals $h_j - c/24$. According to the last Theorem,

$$Z_{A^j}(\tau) = q^{t_j/N} \sum_n dim A_{h+n} q^n.$$

Assuming these q-expansions *are* modular functions on a congruence subgroup, their (common) level will be N.

The associated vector-valued modular function

'Theorem' Let V be a regular VOA. The finite-dimensional space E of holomorphic functions in $\mathcal{H}$ spanned by the partition functions $Z_{A^j}(\tau), j = 1, ..., d$ furnishes a representation of $\Gamma := SL_2(\mathbb{Z})$ in the sense that

$$\text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{then}$$

$$Z_{A^i}\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{j=1}^{d} c_{ij}(\gamma) Z_{A^j}(\tau)$$

Again, the literature contains only a slightly weaker version of this result.

Thus we have a representation of Γ

$$\rho : \Gamma \to GL(E), \quad \gamma \mapsto (c_{ij}(\gamma))$$

Example: holomorphic VOAs

A VOA is *holomorphic* if it is regular and has a *unique* simple module. The last Theorem has the following consequence.

Theorem If V is a holomorphic VOA then $Z_V(q)$ is a modular function of level 1 (perhaps with character).

Examples.

1. Moonshine module $V^\natural$.
2. Self-dual lattice theories $V_L \quad (L = L^*)$.

We think of the last modular-invariance Theorem as follows: a regular VOA furnishes a weight 0 vector-valued modular function

$$F : \mathcal{H} \longrightarrow \mathbb{C}^d$$

$$F(\tau) := \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_d(\tau) \end{pmatrix} \quad (f_i := Z_{A^i})$$

$$F(\gamma\tau) = \rho(\gamma)F(\tau)$$

The previous Theorem on rationality of exponents follows from

Theorem Suppose that $F(\tau)$ is a vector-valued modular function with components $f_i(\tau) = \sum_n a_{ni} q^{h+n/N}$ with Fourier coefficients in $\mathbb{Q}$. Then $h \in \mathbb{Q}$.

Thus far, we have sketched results that say the following:

For a regular VOA V, the following hold:

- finitely many simple V-modules A^j
- convergent q-expansions (poles at the cusps)
  $Z_{A^j}(\tau) = q^{t_j/N} \sum_n dim A_{h+n} q^n$
- vector-valued modular function ${}^tF(\tau) = (Z_{A^1}, \ldots, Z_{A^d})$
  $\rho(\gamma)F(\tau) = F(\gamma\tau), \quad \rho : SL_2(\mathbb{Z}) \to GL_d(\mathbb{C})$

The main conjecture is then reduced to

$$\boxed{ker \rho \supseteq \Gamma(N)}$$

Again, this is open in general. It is not even known if $ker \rho$ has finite index in $SL_2(\mathbb{Z})$.

Approach using MTCs

The idea here is to develop the structure of $\mathbf{V}-\mathbf{mod}$. In particular, there is an abstract tensor product

$$A \boxtimes B$$

and a duality structure

$$A \mapsto A'$$

on the objects. In favorable situations one can expect that $\mathbf{V}-\mathbf{mod}$ has additional properties, for example *finite fusion rules* and a braiding

$$A \boxtimes B \xrightarrow{\cong} B \boxtimes A$$

Such results do hold sometimes, though they are difficult to prove.

For a restricted class of regular VOAs, it has been shown that $\mathbf{V}-\mathbf{mod}$ is a *modular tensor category* (MTC).

This implies that there is a *natural action of the modular group on the space spanned by the simple objects*.

It is known that for any MTC (the connection with VOAs is not relevant here) this action *factors through a congruence subgroup*.

In the case of $\mathbf{V}-\mathbf{mod}$, and with additional assumptions, it is shown that the action differs only marginally from the representation ρ of $SL_2(\mathbb{Z})$ on the space E of partition functions of V. Then it follows that ρ factors through a congruence quotient as required.

The class of regular VOAS V for which this procedure seems to apply consists (roughly) of those with

- $V = \mathbb{C}\mathbf{1} \oplus \ldots$
- $V \cong V'$
- $h_j > 0$ if $A_j \neq V$

This result covers many important cases, but by no means all.

Approach using vector-valued modular forms

Algebraic vvmf conjecture

Assume that

$$F(\tau) := \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_d(\tau) \end{pmatrix} \quad (f_i := \sum_n a_{in} q^{n/N})$$

$$F(\gamma\tau) = \rho(\gamma)F(\tau)$$

is a vector-valued modular function with *algebraic integer Fourier coefficients* a_{in}. Then $\ker\rho \supseteq \Gamma(N)$.

This implies that each f_i is a modular function on $\Gamma(N)$. The conjectured modular-invariance for all regular VOAs, as earlier described, follows immediately.

This conjecture offers a completely different approach to the modular-invariance problem for regular VOAs.

It can be studied dimension-by-dimension, and is known if $d = 1$ (already explained, because then V is holomorphic) and also if $d = 2, 3$ (more difficult). *No* assumptions on the VOA beyond regularity are required. On the other hand, it cannot yet reproduce the results available from the MTC approach.

Unbounded denominators

Evidence that the algebraic vvmf conjecture is hard!

Let $K := \ker \rho$. If K has finite index in $SL_2(\mathbb{Z})$ then each $f_i(\tau)$ is a modular function on K with *algebraic integer* Fourier coefficients.

There is a well-known conjecture, essentially due to Atkin-Swinnerton-Dyer: if $f(\tau)$ is a modular form on a noncongruence subgroup with algebraic Fourier coefficients, then the Fourier coefficients have *unbounded denominators*.

Algebraic vvmf conjecture $\Rightarrow$ A-S-D UBD conjecture

and modular-invariance conjecture for regular VOAs

The algebraic vvmf conjecture (hence the modular-invariance conjecture for regular VOAs), is known if $d \leq 3$. The proof again makes heavy use of the theory of *Fuchsian DEs*.

The idea is to use the theory of vvmfs to find Fuchsian DEs over the j-line whose solution space is spanned by the components of the corresponding vvmf, and whose *monodromy* is the representation ρ.

In this way, one relates the components of the vvmf to *hypergeometric series*.

In particular, this leads to proofs of A-S-D's UBD conjecture in infinitely many new cases.

Finally, what about the case when $K = ker\,\rho$ has *infinite index* in $SL_2(\mathbb{Z})$?

Again one can look for a Fuchsian DE whose monodromy is ρ and whose solutions are the components f_i of the vvmf (perhaps modified in a simple way). This can be achieved.

A-S-D made no conjecture in this case, so what can one do?

In fact, if $d = 2$ then one almost always has K of infinite image. It can be shown that one has unbounded denominators. The lesson is that there is almost certainly a UBD conjecture for algebraic vvmfs that includes A-S-D as a special case.

The *Grothendieck p-curvature conjecture* gives necessary and sufficient conditions that a Fuchsian DE has finite monodromy.

It is an attractive idea that developments related to p-curvature might be used to reduce the algebraic vvmf conjecture to the case of finite monodromy.

THANK YOU!