Framework of Rogers-Ramanujan identities: Lecture 2

Ken Ono (Emory University)

The *Golden Ratio*

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- *The* **golden ratio** *is the algebraic number*

$$\phi := \frac{1+\sqrt{5}}{2} \sim 1.618033989\ldots.$$

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$$x^2 - x - 1 = 0.$$

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- *It is an* **algebraic integral unit**. *It is a root of*

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- *We have that*

$$\phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}.$$

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Question

*Define the q-***continued fraction**

$$R(q) := \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}.$$

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Question

*Define the q-***continued fraction**

$$R(q) := \cfrac{1}{1 + \cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}.$$

Is the evaluation $R(1) = 1/\phi$ a special case of a theory of units?

Ramanujan's first letter to Hardy

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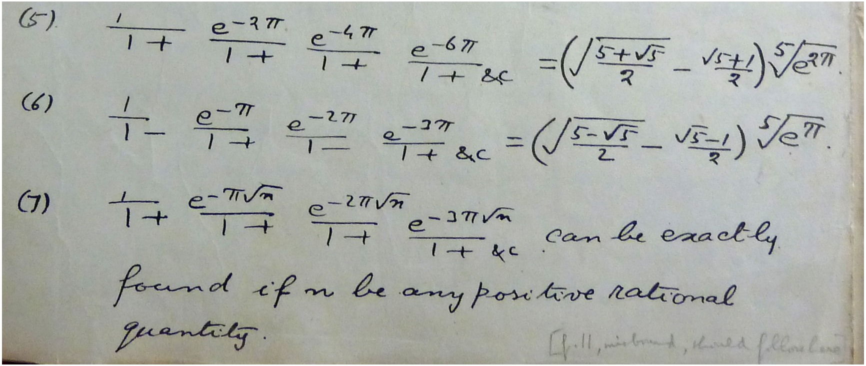

(5) $\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\,\&c = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}$.

(6) $\frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+}\,\&c = \left(\sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}$.

(7) $\frac{1}{1+}\,\frac{e^{-\pi\sqrt{n}}}{1+}\,\frac{e^{-2\pi\sqrt{n}}}{1+}\,\frac{e^{-3\pi\sqrt{n}}}{1+}\,\&c$ can be exactly found if n be any positive rational quantity.

Rogers and Ramanujan

Theorem (Rogers-Ramanujan)

We have that

$$R(q) = \cfrac{1}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} = \prod_{n=0}^{\infty} \frac{(1 - q^{5n+1})(1 - q^{5n+4})}{(1 - q^{5n+2})(1 - q^{5n+3})}.$$

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Theorem (Berndt-Chan-Zhang (1996), Cais-Conrad (2006))

If τ is a CM point, then

$$e^{2\pi i\tau/5} \cdot R(e^{2\pi i\tau})$$

is an **algebraic integral unit.**

Rogers-Ramanujan Identities

Theorem (Rogers, Ramanujan)

We have that

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

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Remark

We have the **ratio** *identity*

$$R(q) = H(q)/G(q).$$

Ubiquity of the RR identities

- Number theory (modular forms and modular curves)
- Conformal field theory
- K-theory
- Kac-Moody Lie algebras
- Knot theory
- Probability theory
- Statistical mechanics
- ...

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Remark (March 10, 2015)

There are 810 papers in **MathSciNet** *about the RR identities!*

Extending RR: Andrews-Gordon Identities

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Theorem (Andrews, 1974)

If $1 \le i \le m+1$, *then*

$$\sum_{r_1 \ge \cdots \ge r_m \ge 0} \frac{q^{r_1^2+\cdots+r_m^2+r_i+\cdots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = \frac{(q^{2m+3};q^{2m+3})_\infty}{(q)_\infty}\cdot\theta(q^i;q^{2m+3}),$$

where

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

and

$$\theta(a;q) := (a;q)_\infty(q/a;q)_\infty.$$

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4. Other Lie theoretic work: Feigin-Frenkel, Milne, Cherednik-Feigin, ...

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Are these hints of a larger framework of conceptual identities:

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Are these hints of a larger framework of conceptual identities:

"Summatory q-series" = "Infinite product modular function"?

Fundamental Problem 2

If so, are there **natural ratios** *which give* **algebraic integral units***?*

Integer Partitions

Definition

A **partition** is a nonincreasing sequence of positive integers

$$\lambda := (\lambda_1, \lambda_2, \dots)$$

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- For positive i we let $m_i :=$ "multiplicity" of size i parts.
- For $n \geq l(\lambda)$ we let $m_0 := n - l(\lambda)$.

Hall-Littlewood symmetric polynomials

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If λ is a partition with $l(\lambda) \leq n$, then let

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and let

$$v_\lambda(q) := \prod_{i=0}^{n} \frac{(q)_{m_i}}{(1-q)^{m_i}}.$$

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The **Hall-Littlewood polynomial** is

$$P_\lambda(x;q) = \frac{1}{v_\lambda(q)} \sum_{w \in S_n} w\left(x^\lambda \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right).$$

Example 1

For $n \geq 1$ we have

$$P_{(2)}(x_1, x_2, \ldots, x_n; q) = \frac{(1-q)^{n-1}}{(q)_{n-1}} \cdot \sum_{w \in S_n} w\left(x_1^2 \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right).$$

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We find that

$$P_{(2)}(x_1; q) = x_1^2$$

$$P_{(2)}(x_1, x_2; q) = x_1^2 + x_2^2 + (1-q)x_1x_2$$

$$P_{(2)}(x_1, x_2, x_3; q) = x_1^2 + x_2^2 + x_3^2 + (1-q)(x_1x_2 + x_1x_3 + x_2x_3)$$

$$\vdots \quad = \quad \vdots$$

Example 1 (Continued)

Letting $x_1 = 1, x_2 = q, x_3 = q^2, \ldots,$ *we obtain*

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Letting $x_1 = 1, x_2 = q, x_3 = q^2, \ldots,$ *we obtain*

$$\begin{aligned} P_{(2)}(1; q) &= 1 \\ P_{(2)}(1, q; q) &= 1 + q \\ P_{(2)}(1, q, q^2; q) &= 1 + q + q^2 \\ \vdots \quad & \quad \vdots \end{aligned}$$

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More generally, for every $n \geq 1$ *we have*

$$P_{(2)}(1, q, q^2, \ldots, q^n; q) = 1 + q + q^2 + \cdots + q^n.$$

Example 1 (Continued)

- *For each* $n \geq 1$ *we have*

$$
\begin{aligned}
&P_{(2)}(x_1, \ldots, x_n; q) \\
&\qquad = \frac{1+q}{2}\left(x_1^2 + \cdots + x_n^2\right) + \frac{1-q}{2}\left(x_1 + \cdots + x_n\right)^2 .
\end{aligned}
$$

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- *Make the identifications*

$$\begin{aligned} (x_1, x_2, \ldots) &\longleftrightarrow (1, q, q^2, \ldots) \\ x_1^r + x_2^r + \cdots + x_n^r &\longleftrightarrow \frac{1}{1-q^r} \end{aligned}$$

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- *Make the identifications*

$$(x_1, x_2, \dots) \quad \longleftrightarrow \quad (1, q, q^2, \dots)$$
$$x_1^r + x_2^r + \cdots + x_n^r \quad \longleftrightarrow \frac{1}{1-q^r}$$

- *This gives us*

$$P_{(2)}(1, q, q^2, \dots; q) = \frac{(1+q)}{2(1-q^2)} + \frac{1-q}{2(1-q)^2} = \frac{1}{1-q}.$$

Example 2

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For $n \geq 2$ *we have*

$$P_{(2,2)}(x_1, x_2, \ldots, x_n; q) = \frac{(1-q)^{n-1}}{(q)_{n-2} \cdot (1-q^2)} \cdot \sum_{w \in S_n} w\left(x_1^2 x_2^2 \prod_{i<j} \frac{x_i - qx_j}{x_i - x_j} \right).$$

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We find that

$$\begin{aligned} P_{(2,2)}(x_1, x_2; q) &= x_1^2 x_2^2 \\ P_{(2,2)}(x_1, x_2, x_3; q) &= x_1^2 x_2^2 + x_2^2 x_3^2 + x_1^2 x_3^2 + \ldots \\ \vdots \quad &= \quad \vdots \end{aligned}$$

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$$\begin{aligned} P_{(2,2)}(1, q; q) &= q^2 \\ P_{(2,2)}(1, q, q^2; q) &= q^2 + q^3 + q^4 \\ P_{(2,2)}(1, q, q^2, q^3; q) &= q^2 + q^3 + 2q^4 + q^5 + q^6 \\ \vdots \quad & \quad \vdots \end{aligned}$$

Example 2 (Continued)

We find that

$$P_{(2,2)}(x_1\dots,x_n;q) = -\frac{q^3-q}{4}(x_1+\cdots+x_n)^2(x_1^2+\cdots+x_n^2)$$
$$+\frac{q^3-3q+2}{24}(x_1+\cdots+x_n)^4+\frac{q^3+q+2}{8}(x_1^2+\cdots+x_n^2)^2$$
$$+\frac{q^3-1}{3}(x_1+\cdots+x_n)(x_1^3+\cdots+x_n^3)-\frac{q^3+q}{4}(x_1^4+\cdots+x_n^4).$$

Example 2 (Continued)

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Arguing as before gives:

$$P_{(2,2)}(1, q, q^2, \dots; q) = \frac{q^2}{(1-q)(1-q^2)}.$$

Hall-Littlewood q-series

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2. *Obtain $P_\lambda(1, q, q^2, \ldots; q^T)$ by replacing*

$$x_1^r + \cdots + x_n^r \quad \longmapsto \quad 1 + q^r + q^{2r} + \cdots = \frac{1}{1 - q^r}.$$

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$$q^{(\sigma+1)|(1^n)|} P_{(2^n)}(1, q, q^2, \ldots ; q) = \frac{q^{n(n+\sigma)}}{(1-q)\cdots(1-q^n)},$$

and so...

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For the partitions $\lambda = (2^n)$, *this procedure gives:*

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and so...

$$\sum_{n=0}^{\infty} q^{(\sigma+1)|(1^n)|} P_{(2^n)}(1, q, q^2, \ldots; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+\sigma)}}{(1-q)\cdots(1-q^n)}.$$

Fundamental Problem 1

"Theorem" (Griffin-O-Warnaar)

There are four triples (a, b, c) *such that* **for all** $m, n \geq 1$ *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{bn+c})$$

$$= \text{"}\textbf{Infinite product modular function}\text{"}.$$

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The $P_\lambda(x_1, \dots; q)$ *are (extended)* **Hall-Littlewood polynomials**.

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Remark

RR identities when $m = n = 1$ *and* $(a, b, c) = (1, 2, -1), (2, 2, -1)$.

q-series representations

If m and n are positive integers, then

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{(\sigma+1)|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q^n)$$
$$= \sum \prod_{i=1}^{2m} \left\{ \frac{q^{\frac{1}{2}(\sigma+1)\mu_i^{(0)}}}{(q^n; q^n)_{\mu_i^{(0)} - \mu_{i+1}^{(0)}}} \prod_{a=1}^{n} q^{\mu_i^{(a)} + n\binom{\mu_i^{(a-1)} - \mu_i^{(a)}}{2}} \begin{bmatrix} \mu_i^{(a-1)} - \mu_{i+1}^{(a)} \\ \mu_i^{(a-1)} - \mu_i^{(a)} \end{bmatrix}_{q^n} \right\},$$

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where the summation **is not worth explaining**.

Representation theoretic interpretation

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- Given an affine Kac-Moody algebra, one has the "principal specialization homomorphism"

$$F_{\mathbb{1}} : \mathbb{C}[[e^{-\alpha_0}, \ldots, e^{-\alpha_n}]] \to \mathbb{C}[[q]], \qquad F_{\mathbb{1}}(e^{-\alpha_i}) = q \quad \forall i \in I.$$

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- Weyl-Kac formula for highest weight modules Λ:

$$F_{\mathbb{1}}\left(e^{-\Lambda}\operatorname{ch} V(\Lambda)\right) = \prod_{\alpha \in \Delta_+^\vee} \left(\frac{1 - q^{\langle \Lambda + \rho, \alpha \rangle}}{1 - q^{\langle \rho, \alpha \rangle}}\right)^{\mathrm{mult}(\alpha)}.$$

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- The "product sides" arise from such formulas.

Fundamental Problem 2

"Theorem" (Griffin-O-Warnaar)

Generalizing the "Folklore Conjecture", in the $A_{2n}^{(2)}$ cases we obtain ratios of CM values that are **algebraic integral units**.

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If $m, n \geq 1$ and $\kappa := 2m + 2n + 1$, then

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$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q^{2n-1})$$

$$= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \cdot \prod_{i=1}^{n} \theta(q^{i+m}; q^\kappa) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa)$$

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Remarks on Theorem 1

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2. If $n = 1$, then we obtain the AG $i = 1, m + 1$ identities.

Easy to use Theorem 1

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Example

If $m = n = 2$, then we obtain **Dyson's favorite**

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3) = \prod_{n=1}^{\infty} \frac{(1 - q^{9n})}{(1 - q^n)},$$

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Example

If $m = n = 2$, then we obtain **Dyson's favorite**

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3) = \prod_{n=1}^{\infty} \frac{(1-q^{9n})}{(1-q^n)},$$

and

$$\sum_{\substack{\lambda \\ \lambda_1 \leq 2}} q^{2|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^3)$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{9n})(1-q^{9n-1})(1-q^{9n-8})}{(1-q^n)(1-q^{9n-4})(1-q^{9n-5})}.$$

Theorem 2 ($C_n^{(1)}$ identities)

If $m, n \geq 1$ *and* $\kappa := 2m + 2n + 2$, *then*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P_{2\lambda}(1, q, q^2, \dots; q^{2n})$$
$$= \frac{(q^2; q^2)_\infty (q^{\kappa/2}; q^{\kappa/2})_\infty (q^\kappa; q^\kappa)_\infty^{n-1}}{(q)_\infty^{n+1}}$$
$$\times \prod_{i=1}^{n} \theta(q^i; q^{\kappa/2}) \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j}; q^\kappa)$$

Theorem 3 ($D_{n+1}^{(2)}$ identities)

If $m \geq 1$, $n \geq 2$, *and* $\kappa := 2m + 2n$, *then*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{2|\lambda|} P_{2\lambda}(1, q, q^2, \ldots ; q^{2n-2}) = \frac{(q^\kappa; q^\kappa)_\infty^n}{(q^2; q^2)_\infty (q)_\infty^{n-1}} \cdot \prod_{1 \leq i < j \leq n} \theta(q^{j-i}, q^{i+j-1}; q^\kappa).$$

Turning to Algebraicity

Turning to Algebraicity

We require the following **renormalizations** (note: $q := e^{2\pi i \tau}$):

$$\Phi_{1a}(m,n;\tau) := q^{\frac{4m^2n^2-4m^2n+2mn^2-3mn}{12\kappa}} \sum_{\lambda_1 \le m} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^{2n-1})$$

$$\Phi_{1b}(m,n;\tau) := q^{\frac{4m^2n^2+2m^2n+2n^2m+3mn}{12\kappa}} \sum_{\lambda_1 \le m} q^{2|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^{2n-1})$$

$$\Phi_{2}(m,n;\tau) := q^{\frac{4m^2n^2+2mn^2-mn-m^2-m}{12\kappa}} \sum_{\lambda_1 \le m} q^{|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^{2n})$$

$$\Phi_{3}(m,n;\tau) := q^{\frac{4m^2n^2-2m^2n+2mn^2+mn-m}{12\kappa}} \sum_{\lambda_1 \le m} q^{2|\lambda|} P_{2\lambda}(1,q,q^2,\dots;q^{2n-2}).$$

CM values

CM values

Theorem 4

If $\kappa\tau$ is a CM point with discriminant $-D < 0$, then the CM value $\Phi_*(m, n; \tau)$ is algebraic.

Algebraic integral units

Theorem (Berndt-Chan-Zhang, Cais-Conrad)

If τ is a CM point, then $q^{1/5}R(q) = \Phi_{1a}(1, 1; \tau)/\Phi_{1b}(1, 1; \tau)$ is an algebraic integral unit.

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Question

Do other ratios of Φ_* have CM values with unit ratios?

Integrality properties

Theorem 5

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Remarks

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Remarks

1. *Theorem 5 (3) is the $q^{1/5}R(q)$ result when $m = n = 1$.*
2. **No other** *ratios generically give units.*

Example when $m = n = 2$

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- For $\tau = i/3$ the first 100 terms give:

$$\Phi_{1a}(2,2;i/3) = 0.577350\cdots \stackrel{?}{=} \frac{1}{\sqrt{3}}$$
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- They **are not** algebraic integers, but are roots of:

$$3x^2 - 1$$
$$3^9x^{18} - 3^7\cdot 37x^{12} - 2\cdot 3^9x^9 + 2^3\cdot 3^4\cdot 17x^6 - 2\cdot 3^5x^3 - 1.$$

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- By Theorem 5 (2), **both** $\sqrt{3}\Phi_{1*}(2,2;i/3)$ are integral units.

Example when $m = n = 2$ continued.

- Which gives Theorem 5 (3) that

$$\Phi_{1a}(2,2;i/3)/\Phi_{1b}(2,2;i/3) = 4.60627\ldots$$

is an algebraic integral unit.

Example when $m = n = 2$ continued.

- Which gives Theorem 5 (3) that

$$\Phi_{1a}(2,2;i/3)/\Phi_{1b}(2,2;i/3) = 4.60627\ldots$$

  is an algebraic integral unit.

- Indeed, $\Phi_{1a}(2,2;i/3)/\Phi_{1b}(2,2;i/3)$ is a root of

$$x^{18} - 102x^{15} + 420x^{12} - 304x^{9} - 93x^{6} + 6x^{3} + 1.$$

Classical proof of RR

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Theorem (G. N. Watson (1929))

$$\frac{(aq, aq/bc)_N}{(aq/b, aq/c)_N} \sum_{r=0}^{N} \frac{(b, c, aq/de, q^{-N})_r}{(q, aq/d, aq/e, bcq^{-N}/a)_r} q^r$$

$$= \sum_{r=0}^{N} \frac{1 - aq^{2r}}{1 - a} \cdot \frac{(a, b, c, d, e, q^{-N})_r}{(q, aq/b, aq/c, aq/d, aq/e)_r} \cdot \left(\frac{a^2 q^{N+2}}{bcde}\right)^r .$$

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Corollary (Rogers-Selberg Identity)

$$\sum_{r=0}^{\infty} \frac{a^r q^{r^2}}{(q)_r} = \frac{1}{(aq)_\infty} \sum_{r=0}^{\infty} \frac{1-aq^{2r}}{1-a} \cdot \frac{(a)_r}{(q)_r} \cdot (-1)^r a^{2r} q^{5\binom{r}{2}+2r}.$$

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- Letting $a = 1, q$ on the LHS gives RR.
- What is the RHS when $a = 1, q$?

Proof of the RR identities continued

Lemma (Jacobi Triple Product)

$$\sum_{r=-\infty}^{\infty} (-1)^r x^r q^{\binom{r}{2}} = (q)_\infty \cdot \theta(x;q),$$

Proof of the RR identities continued

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- Rogers-Selberg + JTP $\implies$ RR. $\square$

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4. Resulting in a higher "Rogers-Selberg" identity.

5. If $a = 1, q$, then JTP essentially gives AG. $\square$

Proving Theorem 1-3

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"Theorem" (Bartlett-Warnaar (2013))

There is an Andrews-style "crazier" transformation, arising from the C_n root system, where

$$a \quad \longleftrightarrow \quad (x_1, x_2, \ldots, x_n).$$

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There is an Andrews-style "crazier" transformation, arising from the C_n root system, where

$$a \quad \longleftrightarrow \quad (x_1, x_2, \ldots, x_n).$$

Remark

Their transformation laws make use of

$$\Delta_{\mathbb{C}}(x) := \prod_{i=1}^{n}(1 - x_i^2) \prod_{1 \le i < j \le n} (x_i - x_j)(x_i x_j - 1).$$

Bartlett-Warnaar Transformation Law

Theorem 4.2 (C_n Andrews transformation). *For m a nonnegative integer and $N \in \mathbb{Z}_+^n$,*

$$\sum_{0 \subseteq r \subseteq N} \frac{\Delta_C(xq^r)}{\Delta_C(x)} \prod_{i=1}^{n} \left[\prod_{\ell=1}^{m+1} \frac{(b_\ell x_i, c_\ell x_i)_{r_i}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i}} \left(\frac{q}{b_\ell c_\ell}\right)^{r_i} \times \prod_{j=1}^{n} \frac{(q^{-N_j} x_i/x_j, x_i x_j)_{r_i}}{(qx_i/x_j, q^{N_j+1} x_i x_j)_{r_i}} q^{N_j r_i} \right] \tag{4.3}$$

$$= \prod_{i,j=1}^{n} (qx_i x_j)_{N_i} \prod_{1 \le i < j \le n} \frac{1}{(qx_i x_j)_{N_i+N_j}}$$

$$\times \sum_{r^{(1)},\dots,r^{(m)} \in \mathbb{Z}_+^n} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{N_i}}{(qx_i/x_j)_{N_i - r_j^{(1)}}} \prod_{\ell=1}^{m} f^{(0)}_{r^{(\ell)}, r^{(\ell+1)}}(x;q)$$

$$\times \prod_{\ell=1}^{m+1} \left[(q/b_\ell c_\ell)_{|r^{(\ell-1)}|-|r^{(\ell)}|} \left(\frac{q}{b_\ell c_\ell}\right)^{|r^{(\ell)}|} \prod_{i=1}^{n} \frac{(b_\ell x_i, c_\ell x_i)_{r_i^{(\ell)}}}{(qx_i/b_\ell, qx_i/c_\ell)_{r_i^{(\ell-1)}}} \right],$$

where $r^{(0)} := N$ and $r^{(m+1)} := 0$.

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- Make use of the added flexibility.
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- Analyze the RHS....using definition of Hall-Littlewood polynomials.

Theorem (Higher Rogers-Selberg Identity)

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{|\lambda|} P'_{2\lambda}(x;q) = L_m^{(0)}(x;q),$$

where

$$L_m^{(0)}(x;q) := \sum_{r \in \mathbb{Z}_+^n} \frac{\Delta_{\mathbb{C}}(xq^r)}{\Delta_{\mathbb{C}}(x)} \times \prod_{i=1}^{n} x_i^{2(m+1)r_i} q^{(m+1)r_i^2 + n\binom{r_i}{2}} \cdot \prod_{i,j=1}^{n} \left(-\frac{x_i}{x_j}\right)^{r_i} \frac{(x_i x_j)_{r_i}}{(q x_i/x_j)_{r_i}}.$$

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- Many pages of reformulations involving Macdonald identities for

$$D_{n+1}^{(2)}, \quad B_n^{(1)}, \quad D_n^{(1)},$$

Weyl-Kac denominator formulas, and of course JTP. □

Turning to algebraic properties

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Definition

If $\mathbf{B}_2(x)$ is the 2nd Bernoulli polynomial, $e(x) := \mathrm{e}^{2\pi i x}$ and $a := (a_1, a_2) \in \mathbb{Q}^2$, then the **Siegel function** g_a is defined by

$$\begin{aligned} g_a(\tau) := &-q^{\frac{1}{2}\mathbf{B}_2(a_1)} e(a_2(a_1-1)/2) \\ &\times \prod_{n=1}^{\infty} (1-q^{n-1+a_1}e(a_2))(1-q^{n-a_1}e(-a_2)), \end{aligned}$$

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Theorem (Klein)

If $a \in \mathbb{Z}^2/N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then

$$g_a^{12}(\gamma\tau) = g_{a\gamma}^{12}(\tau).$$

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If $m, n \geq 1$, then the following are true:

(1a) *If $\kappa = \kappa_1(m,n) = 2m + 2n + 1$, then we have*

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(1b) *...and so on...*

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- Kubert-Lang extended to "products" of Siegel functions.
- The classical theory of **complex multiplication.**
- Analytic number theory to obtain single Galois orbits. □

Our results...

"Theorem" (Griffin-O-Warnaar)

There are four triples (a, b, c) *such that* **for all** $m, n \geq 1$ *we have*

$$\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{a|\lambda|} P_{2\lambda}(1, q, q^2, \ldots; q^{bn+c})$$

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1. *RR identities when* $m = n = 1$ *in Theorem 1.*
2. *Arise as* **specialized characters** *of Kac-Moody Lie algebras.*

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Remark

Letting $m = n = 1$ gives Berndt-Chan-Zhang and Cais-Conrad.
And $\tau = i$ gives Ramanujan's evaluation:

$$e^{-2\pi/5} \cdot R(e^{-2\pi}) = \frac{\Phi_{1a}(1, 1; i)}{\Phi_{1b}(1, 1; i)} = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}.$$