Mock theta functions
and
representation theory of
affine Lie superalgebras and superconformal algebras
$\sim$ Lecture 2 $\sim$

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"Characters of Representations and Modular Forms"

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Weyl-Kac type character formula for partially integrable representations of affine Lie superalgebras:

$$\mathrm{ch}_{L(\Lambda)} \; = \; \frac{1}{R} \sum_{w \in W^{\sharp}} \varepsilon(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} \left(1 + e^{-\beta_i}\right)} \right)$$

$$\text{where} \qquad \begin{cases} \beta_i & \in \quad \Pi \quad = \{\text{simple roots}\} \\ (\Lambda+\rho|\beta_i) & = \quad 0 \\ (\beta_i|\beta_j) & = \quad 0 \\ \{\beta_i\}_{i=1,\cdots,n} & : \quad \text{maximal} \qquad (n \; := \; \text{atypicality}) \end{cases}$$

<u>super-character:</u>

$$\mathrm{ch}^{(-)}_{L(\Lambda)} \;:=\; \frac{1}{R^{(-)}} \sum_{w \in W^\sharp} \varepsilon^{(-)}(w)\, w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} \left(1 - e^{-\beta_i}\right)} \right)$$

where

$$\varepsilon^{(-)}(r_\alpha) \;:=\; \begin{cases} 1 & \text{if} \quad \alpha/2 \;=\; \text{root} \\ -1 & \text{if} \quad \alpha/2 \;\neq\; \text{root} \end{cases}$$

$$R^{(\pm)} \;:=\; e^{\rho} \; \frac{\prod_{\alpha \in \Delta_+^{\text{even}}} (1 - e^{-\alpha})^{\text{mult}\alpha}}{\prod_{\alpha \in \Delta_+^{\text{odd}}} (1 \pm e^{-\alpha})^{\text{mult}\alpha}} \quad : \quad \text{(super)denominator of } \mathfrak{g}$$

Example. In the case $\widehat{sl}(2|1)$:

$$R^{(-)} \cdot \mathrm{ch}^{(-)}_{L((m-1)\Lambda_0)} \quad = \quad e^{2\pi imt}\left\{\sum_{j\in\mathbf{Z}} \frac{e^{2\pi imj(z_1+z_2)}q^{mj^2}}{1-e^{2\pi iz_1}q^j} - \sum_{j\in\mathbf{Z}} \frac{e^{-2\pi imj(z_1+z_2)}q^{mj^2}}{1-e^{-2\pi iz_2}q^j}\right\}$$

$(m \in \mathbf{N})$

$$\begin{array}{cc} \| \ \textit{put} & \| \ \textit{put} \\ \Phi_1^{[m]} & \Phi_2^{[m]} \end{array}$$

Basic mock theta functions $\Phi_1^{(\pm)[m;s]}$

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2) \overset{def}{:=} \sum_{j\in\mathbf{Z}} (\pm 1)^j \frac{e^{2\pi i m j(z_1+z_2)+2\pi i s z_1} q^{mj^2+sj}}{1-e^{2\pi i z_1} q^j} \qquad \begin{pmatrix} m & \in & \frac{1}{2}\mathbf{N} \\ s & \in & \frac{1}{2}\mathbf{Z} \end{pmatrix}$$

$$\Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2, t) \overset{def}{:=} e^{2\pi i m t}\, \Phi_1^{(\pm)[m;s]}(\tau, z_1, z_2)$$

$$\left\{ \begin{array}{l} \text{quasi-elliptic transformation properties} \\ \\ \Phi_1^{(\pm)[m;s]} - \Phi_1^{(\pm)[m;s']}|S \quad : \quad \text{holomorphic} \end{array} \right. \quad + \quad \text{Zwegers' function } R_{j;m}^{(\pm)}$$

$$\Downarrow$$

$$\widetilde{\Phi}_1^{(\pm)[m;s]} \quad : \quad \text{(non-holomorphic) modular form}$$

Theorem. Let $m \in \frac{1}{2}\mathbf{N}$, $s, s' \in \frac{1}{2}\mathbf{Z}$.

1) S-transformation: if $s \in \mathbf{Z}$ and $s' \in \frac{1}{2} + \mathbf{Z}$,

- $\widetilde{\Phi}_1^{(+)[m;s]}\big|_S = \widetilde{\Phi}_1^{(+)[m;s]}$
- $\widetilde{\Phi}_1^{(-)[m;s]}\big|_S = \widetilde{\Phi}_1^{(+)[m;s']}$
- $\widetilde{\Phi}_1^{(+)[m;s']}\big|_S = \widetilde{\Phi}_1^{(-)[m;s]}$
- $\widetilde{\Phi}_1^{(-)[m;s']}\big|_S = \widetilde{\Phi}_1^{(-)[m;s']}$

2) T-transformation:

$$\widetilde{\Phi}_1^{(\pm)[m;s]}\big|_T = \begin{cases} \widetilde{\Phi}_1^{(\pm)[m;s]} & \text{if} \quad m+s \in \mathbf{Z} \\ \widetilde{\Phi}_1^{(\mp)[m;s]} & \text{if} \quad m+s \in \frac{1}{2}+\mathbf{Z} \end{cases}$$

3) $s - s' \in \mathbf{Z} \implies \widetilde{\Phi}^{(\pm)[m;s]} = \widetilde{\Phi}^{(\pm)[m;s']}$

Elliptic transformation properties for $\widetilde{\Phi}^{(\pm)[m;s]}_1$ $\quad (m \in \frac{1}{2}\mathbf{N}, \; s \in \frac{1}{2}\mathbf{Z})$:

Theorem. Let $\begin{cases} m & \in \frac{1}{2}\mathbf{N} \\ a, \, b & \in \mathbf{Z} \\ a+b & \in 2\mathbf{Z} \end{cases}$ or $\begin{cases} m & \in \mathbf{N} \\ a, \, b & \in \mathbf{Z} \end{cases}$. Then

- $\widetilde{\Phi}^{(\pm)[m;s]}_1(\tau, \, z_1+a, \, z_2+b, \, t) \; = \; e^{2\pi isa} \, \widetilde{\Phi}^{(\pm)[m;s]}_1(\tau, z_1, z_2, t)$

- $\widetilde{\Phi}^{(\pm)[m;s]}_1(\tau, \, z_1+a\tau, \, z_2+b\tau, \, t) \; = \; (\pm 1)^a \, e^{-2\pi im(bz_1+az_2)} \, q^{-mab} \, \widetilde{\Phi}^{(\pm)[m;s]}_1(\tau, z_1, z_2, t)$

We now consider mock theta functions

in general setting.

Supercharacter in general case:

$$R^{(-)}\mathrm{ch}^{(-)}_{L(\Lambda)} = \sum_{w \in W^{\sharp}} \varepsilon^{(-)}(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} \left(1 - e^{-\beta_i}\right)} \right)$$

where

$$\begin{cases} \beta_i \in \Pi = \{\text{simple roots}\} \\ (\Lambda + \rho | \beta_i) = 0 \\ (\beta_i | \beta_j) = 0 \\ \{\beta_i\}_{i=1,\cdots,n} \ : \ \text{maximal} \end{cases}$$

and

$$\varepsilon^{(-)}(r_\alpha) := \begin{cases} 1 & \text{if } \alpha/2 \text{ is a root} \\ -1 & \text{otherwise} \end{cases} \qquad (\alpha \ : \ \text{even root})$$

Normalized supercharacter:

$$R^{(-)}\mathrm{ch}^{(-)}_{\Lambda} \overset{def}{:=} q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}} \; R^{(-)}\mathrm{ch}^{(-)}_{L(\Lambda)}$$

$$= q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}} \sum_{w\in W^{\sharp}} \varepsilon^{(-)}(w)w\left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n}\left(1-e^{-\beta_i}\right)}\right)$$

$$= \sum_{w\in \overline{W}^{\sharp}} \varepsilon^{(-)}(w)w\left(\sum_{\alpha\in M^{\sharp}} q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}}\varepsilon^{(-)}(t_\alpha)t_\alpha\left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n}\left(1-e^{-\beta_i}\right)}\right)\right)$$

where

$$t_\alpha(\lambda) := \lambda + (\lambda|\delta)\alpha - \left\{\frac{|\alpha|^2}{2}(\lambda|\delta) + (\lambda|\alpha)\right\}\delta$$

Note: For an even root α, $t_{\alpha^\vee} = r_{\delta-\alpha}\, r_\alpha$

Then

$$\varepsilon(t_{\alpha^\vee}) = \underbrace{\varepsilon(r_{\delta-\alpha})}_{\substack{\| \\ -1}}\ \underbrace{\varepsilon(r_\alpha)}_{\substack{\| \\ -1}} = 1$$

$$\varepsilon^{(-)}(t_{\alpha^\vee}) = \underbrace{\varepsilon^{(-)}(r_{\delta-\alpha})}_{\substack{\| \\ -1}}\ \underbrace{\varepsilon^{(-)}(r_\alpha)}_{\substack{\| \\ \begin{cases} 1 \\ -1 \end{cases}}} = \begin{cases} -1 & \text{if} \quad \alpha/2 = \text{root} \\ 1 & \text{if} \quad \alpha/2 \neq \text{root} \end{cases}$$

For simplicity, consider <u>the case $\varepsilon^{(-)} = \varepsilon$</u> :

$$R^{(-)}\mathrm{ch}_\Lambda^{(-)} = \sum_{w \in \overline{W}^\sharp} \varepsilon(w) w \left(q^{\frac{|\Lambda+\rho|^2}{2(K+h^\vee)}} \sum_{\alpha \in M^\sharp} t_\alpha \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n} (1-e^{-\beta_i})} \right) \right)$$

$$= \sum_{i \in I} \varepsilon(g_i) g_i \sum_{w \in \overline{W}^!} \varepsilon(w) \underbrace{\left(q^{\frac{|\Lambda+\rho|^2}{2(K+h^\vee)}} \sum_{\alpha \in M^\sharp} t_\alpha \left(\frac{e^{w(\Lambda+\rho)}}{\prod_{i=1}^{n} (1-e^{-\beta_i})} \right) \right)}_{\substack{|| \ put \\ F_{w(\Lambda+\rho)}}}$$

where

$$\overline{W}^! := \{ w \in \overline{W}^\sharp \ ; \ w(\beta_j) = \beta_j \}$$

$$\{g_i\}_{i \in I} \quad : \quad \text{a set of representatives of } \overline{W}^\sharp / \overline{W}^!$$

$$\text{i.e,} \qquad \overline{W}^\sharp = \bigcup_{i \in I} g_i \overline{W}^!$$

Want to find a modification of $F_{w(\Lambda+\rho)}$ to a modular form.

For simplicity, put $\lambda := w(\Lambda+\rho)$ and consider the function

$$F_\lambda = q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{\alpha\in M^\sharp} t_\alpha \left(\frac{e^\lambda}{\prod_{i=1}^{n}(1-e^{-\beta_i})} \right)$$

Assume that $^\exists\{\gamma_i\}_{i=1,\cdots,m}$ such that

- $\{\gamma_i\}_{i=1,\cdots,m}$: $\mathbf{Z}$-basis of $M^\sharp$
- $(\beta_i|\gamma_j) = -\delta_{i,j}$
- $\{\beta_i\}, \ \{\gamma_i\}$: a basis of $\overline{\mathfrak{h}}$ so $\dim\overline{\mathfrak{h}} = m+n$

and put

$$\widetilde{\gamma}_i := \gamma_i + \sum_{j=1}^{\min\{i-1,n\}} (\gamma_i|\gamma_j)\beta_j \qquad (1 \leqq i \leqq m)$$

$$M := \bigoplus_{i=n+1}^{m} \mathbf{Z}\,\widetilde{\gamma}_i$$

Note.

- $$(\widetilde{\gamma}_i|\beta_k) \;=\; \underbrace{(\gamma_i|\beta_k)}_{\substack{\| \\ -\delta_{i,k}}} + \sum_j (\gamma_i|\gamma_j)\underbrace{(\beta_j|\beta_k)}_{\substack{\| \\ 0}} \;=\; -\delta_{i,k}$$

- $k \leqq \min\{i-1,\, n\} \quad \Longrightarrow$

$$(\widetilde{\gamma}_i|\gamma_k) \;=\; (\gamma_i|\gamma_k) + \underbrace{\sum_{j=1}^{\min\{i-1,n\}} (\gamma_i|\gamma_j)\overbrace{(\beta_j|\gamma_k)}^{\substack{-\delta_{j,k} \\ \|}}}_{\substack{\| \\ -(\gamma_i|\gamma_k)}} \;=\; 0$$

- In particular

$$\left.\begin{array}{rcccl} n+1 & \leqq & i & \leqq & m \\ 1 & \leqq & k & \leqq & n \end{array}\right\} \quad \Longrightarrow \quad (\widetilde{\gamma}_i|\gamma_k) \;=\; 0$$

Theorem 4.

1) λ is written in the form:

$$\lambda = \underbrace{(K+h^\vee)\Lambda_0 + \sum_{i=n+1}^{m} k_i \widetilde{\gamma}_i}_{\substack{\| \ put \\ \lambda_M}} + \sum_{i=1}^{n} \underbrace{k_i}_{\substack{\| \\ -(\lambda|\gamma_i)}} \beta_i$$

2) $$\widetilde{F}_\lambda = \Theta_{\lambda_M} \times \prod_{p=1}^{n} \widetilde{\Phi}_1^{[\frac{K+h^\vee}{2}|\gamma_p|^2;\,(\lambda|\gamma_p)]}\Big(\tau,\ -\beta_p,\ \beta_p + \frac{2}{|\gamma_p|^2}\widetilde{\gamma}_p,\ 0\Big)$$

where

$$\Theta_{\lambda_M} := \sum_{\alpha\in M} e^{\lambda+(K+h^\vee)\alpha}\, q^{\frac{1}{K+h^\vee}|\lambda+(K+h^\vee)\alpha|^2}$$

(classical theta function over the lattice M)

3) In general case where $\varepsilon^{(-)}(t_\alpha) = \pm 1$ occurs; i.e,

$$F_\lambda \ := \ q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{\alpha \in M^\sharp} (\pm 1)^{|\alpha|^2} t_\alpha \left(\frac{e^\lambda}{\prod_{i=1}^{n}(1-e^{-\beta_i})} \right)$$

its modification $\widetilde{F}_\lambda$ is given by the signed theta functions

$$\widetilde{F}_\lambda \ = \ \Theta^{(\pm)}_{\lambda_M} \times \prod_{p=1}^{n} \widetilde{\Phi}_1^{(\pm)[\frac{K+h^\vee}{2}|\gamma_p|^2;\,(\lambda|\gamma_p)]} \Big(\tau, \ -\beta_p, \ \beta_p + \frac{2}{|\gamma_p|^2}\widetilde{\gamma}_p, \ 0 \Big)$$

where

$$\Theta^{(\pm)}_{\lambda_M} \ := \ \sum_{\alpha \in M} (\pm 1)^{|\alpha|^2} e^{\lambda_M + (K+h^\vee)\alpha} \, q^{\frac{1}{K+h^\vee}|\lambda_M + (K+h^\vee)\alpha|^2}$$

$\square$

<u>**Proof of Theorem 4. 1)**</u> **:** Since $\{\beta_i\}$, $\{\gamma_i\}$ are basis of $\overline{\mathfrak{h}}$, we can put

$$\lambda \;=\; (K+h^\vee)\Lambda_0 + \sum_{j=1}^{m} a_j\gamma_j + \sum_{j=1}^{n} b_j\beta_j$$

Then

$$\underset{(1 \leqq i \leqq n)}{0 \;=\; (\lambda|\beta_i)} \;=\; \sum_{j=1}^{m} a_j \underbrace{(\gamma_j|\beta_i)}_{\substack{\| \\ -\delta_{i,j}}} + \sum_{j=1}^{n} b_j \underbrace{(\beta_j|\beta_i)}_{0} \;=\; -a_i$$

so

$$\lambda = (K+h^\vee)\Lambda_0 + \sum_{i=n+1}^{m} a_i \underbrace{\gamma_i}_{\substack{\| \\ \widetilde{\gamma}_i - \sum_j (\gamma_i|\gamma_j)\beta_j}} + \sum_{j=1}^{n} b_j\beta_j \;=\; (K+h^\vee)\Lambda_0 + \sum_{i=n+1}^{m} a_i\widetilde{\gamma}_i + \sum_{i=1}^{n} k_i\beta_i$$

And

$$\underset{(1 \leqq j \leqq n)}{(\lambda|\gamma_j)} \;=\; (K+h^\vee)\underbrace{(\Lambda_0|\gamma_j)}_{0} + \sum_{i=n+1}^{m} a_i \underbrace{(\widetilde{\gamma}_i|\gamma_j)}_{0} + \sum_{i=1}^{n} k_i \underbrace{(\beta_i|\gamma_j)}_{\substack{\| \\ -\delta_{i,j}}} \;=\; -k_j$$

$\square$

<u>**Proof of Theorem 4. 2)**</u> **:** Let $M^\sharp \ni \alpha = \sum_{i=1}^{m} j_i \gamma_i$; then

- $$(\alpha|\beta_p) = \sum_{i=1}^{m} j_i \underbrace{(\gamma_i|\beta_p)}_{\substack{\| \\ -\delta_{i,p}}} = -j_p$$

- $$t_\alpha \beta_p = \beta_p - \underbrace{(\alpha|\beta_p)}_{\substack{\| \\ -j_p}} \delta = \beta_p + j_p \delta$$

- $$t_\alpha(\lambda) = \lambda + (K+h^\vee)\alpha - \left\{ \frac{K+h^\vee}{2} |\alpha|^2 + (\lambda|\alpha) \right\} \delta$$

$$= \lambda_M - \sum_{i=1}^{n} (\lambda|\gamma_i)\beta_i + (K+h^\vee) \sum_{i=1}^{m} j_i \gamma_i - \left\{ \frac{K+h^\vee}{2} \Big| \sum_{i=1}^{m} j_i \gamma_i \Big|^2 + \sum_{i=1}^{m} j_i (\lambda|\gamma_i) \right\} \delta$$

Then

$$F_\lambda = q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{\alpha\in M^\sharp} t_\alpha\left(\frac{e^\lambda}{\prod_{p=1}^{n}(1-e^{-\beta_p})}\right)$$

$$= e^{\lambda_M} q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{j_1,\cdots,j_m\in\mathbf{Z}} \frac{e^{-\sum_{i=1}^{n}(\lambda|\gamma_i)\beta_i+(K+h^\vee)\sum_{i=1}^{m}j_i\gamma_i}\, q^{\frac{K+h^\vee}{2}\left|\sum_{i=1}^{m}j_i\gamma_i\right|^2+\sum_{i=1}^{m}j_i(\lambda|\gamma_i)}}{\prod_{p=1}^{n}(1-e^{-\beta_p}\,q^{j_p})}$$

$$= e^{\lambda_M} q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{j_2,\cdots,j_m\in\mathbf{Z}} \frac{e^{-\sum_{i=2}^{n}(\lambda|\gamma_i)\beta_i+(K+h^\vee)\sum_{i=2}^{m}j_i\gamma_i}\, q^{\frac{K+h^\vee}{2}\left|\sum_{i=2}^{m}j_i\gamma_i\right|^2+\sum_{i=2}^{m}j_i(\lambda|\gamma_i)}}{\prod_{p=2}^{n}(1-e^{-\beta_p}\,q^{j_p})}$$

$$\times \underbrace{\sum_{j_1\in\mathbf{Z}} \frac{e^{-(\lambda|\gamma_1)\beta_1+(K+h^\vee)j_1\gamma_1}\, q^{\frac{K+h^\vee}{2}j_1^2|\gamma_1|^2+(K+h^\vee)j_1\left(\gamma_1\middle|\sum_{i=2}^{m}j_i\gamma_i\right)+j_1(\lambda|\gamma_1)}}{1-e^{-\beta_1}\,q^{j_1}}}_{\substack{\| \\ (A)_1}}$$

Compute $(A)_1$:

$$(A)_1 = \sum_{j_1 \in \mathbf{Z}} \frac{e^{-(\lambda|\gamma_1)\beta_1} \overbrace{e^{(K+h^\vee)j_1\gamma_1} q^{(K+h^\vee)j_1\left(\gamma_1 \middle| \sum_{i=2}^{m} j_i\gamma_i\right)}}^{\parallel \atop e^{\frac{K+h^\vee}{2}|\gamma_1|^2 j_1 \left\{\frac{2}{|\gamma_1|^2}\gamma_1 + \frac{2\tau}{|\gamma_1|^2}\left(\gamma_1 \middle| \sum_{i=2}^{m} j_i\gamma_i\right)\right\}}} q^{\frac{K+h^\vee}{2}j_1^2|\gamma_1|^2 + j_1(\lambda|\gamma_1)}}{1 - e^{-\beta_1} q^{j_1}}$$

$$= \Phi_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;\,(\lambda|\gamma_1)\right]}\left(\tau,\ -\beta_1,\ \beta_1 + \frac{2}{|\gamma_1|^2}\gamma_1 + \frac{2\tau}{|\gamma_1|^2}\left(\gamma_1 \middle| \sum_{i=2}^{m} j_i\gamma_i\right)\right)$$

As the 1st step of modification, we replace Φ_1 by $\widetilde{\Phi}_1$ and put

$$
{}^1\widetilde{F}_\lambda \;:=\; e^{\lambda_M}\, q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{j_2,\cdots,j_m \in \mathbf{Z}} \frac{e^{-\sum\limits_{i=2}^{n}(\lambda|\gamma_i)\beta_i + (K+h^\vee)\sum\limits_{i=2}^{m} j_i\gamma_i}\, q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=2}^{m} j_i\gamma_i\right|^2 + \sum\limits_{i=2}^{m} j_i(\lambda|\gamma_i)}}{\prod\limits_{p=2}^{n}(1-e^{-\beta_p}\,q^{j_p})}
$$

$$
\times\; \underbrace{\widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\left(\tau,\;\; -\beta_1,\;\; \beta_1 + \frac{2}{|\gamma_1|^2}\gamma_1 + \frac{2\tau}{|\gamma_1|^2}\Big(\gamma_1\Big|\sum_{i=2}^{m} j_i\gamma_i\Big)\right)}_{\begin{array}{c}\|\\ e^{(K+h^\vee)\beta_1\left(\gamma_1\middle|\sum\limits_{i=2}^{m} j_i\gamma_i\right)}\,\widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\left(\tau,\;\; -\beta_1,\;\; \beta_1+\frac{2}{|\gamma_1|^2}\gamma_1\right)\end{array}}
$$

$$
=\; e^{\lambda_M} q^{\frac{|\lambda|^2}{2(K+h^\vee)}}\;\; \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\left(\tau,\;\; -\beta_1,\;\; \beta_1+\frac{2}{|\gamma_1|^2}\gamma_1\right)
$$

$$
\times \sum_{j_2,\cdots,j_m \in \mathbf{Z}} \frac{e^{-\sum\limits_{i=2}^{n}(\lambda|\gamma_i)\beta_i + (K+h^\vee)\sum\limits_{i=2}^{m} j_i[\gamma_i + (\gamma_i|\gamma_1)\beta_1]}\, q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=2}^{m} j_i\gamma_i\right|^2 + \sum\limits_{i=2}^{m} j_i(\lambda|\gamma_i)}}{\prod\limits_{p=2}^{n}(1-e^{-\beta_p}\,q^{j_p})}
$$

$$= e^{\lambda_M} q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\left(\tau, -\beta_1, \beta_1 + \frac{2}{|\gamma_1|^2}\gamma_1\right)$$

$$\times \sum_{j_3,\cdots,j_m \in \mathbf{Z}} \frac{e^{-\sum_{i=3}^{n}(\lambda|\gamma_i)\beta_i + (K+h^\vee)\sum_{i=3}^{m} j_i[\gamma_i + (\gamma_i|\gamma_1)\beta_1]} q^{\frac{K+h^\vee}{2}\left|\sum_{i=3}^{m} j_i\gamma_i\right|^2 + \sum_{i=3}^{m} j_i(\lambda|\gamma_i)}}{\prod_{p=3}^{n}(1-e^{-\beta_p} q^{j_p})}$$

$$\times \underbrace{\sum_{j_2 \in \mathbf{Z}} \frac{e^{-(\lambda|\gamma_2)\beta_2 + (K+h^\vee)j_2[\gamma_2 + (\gamma_2|\gamma_1)\beta_1]} q^{\frac{K+h^\vee}{2}j_2^2|\gamma_2|^2 + (K+h^\vee)j_2\left(\gamma_2\middle|\sum_{i=3}^{m} j_i\gamma_i\right) + j_2(\lambda|\gamma_2)}}{1-e^{-\beta_2} q^{j_2}}}_{\substack{\| \\ (A)_2}}$$

Compute $(A)_2$: putting $\widetilde{\gamma}_2 := \gamma_2 + (\gamma_2|\gamma_1)\beta_1$

$$(A)_2 = \sum_{j_2 \in \mathbf{Z}} \frac{e^{-(\lambda|\gamma_2)\beta_2} \overbrace{e^{(K+h^\vee)j_2\widetilde{\gamma}_2} q^{(K+h^\vee)j_2\left(\gamma_2 \middle| \sum_{i=3}^m j_i\gamma_i\right)}}^{\substack{e^{\frac{K+h^\vee}{2}|\gamma_2|^2 j_2\left\{\frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2 + \frac{2\tau}{|\gamma_2|^2}\left(\gamma_2 \middle| \sum_{i=3}^m j_i\gamma_i\right)\right\}} \\ \|}} q^{\frac{K+h^\vee}{2}j_2^2|\gamma_2|^2 + j_2(\lambda|\gamma_2)}}{1 - e^{-\beta_2} q^{j_2}}$$

$$= \Phi_1^{\left[\frac{K+h^\vee}{2}|\gamma_2|^2;\,(\lambda|\gamma_2)\right]}\left(\tau,\ -\beta_2,\ \beta_2 + \frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2 + \frac{2\tau}{|\gamma_2|^2}\left(\gamma_2 \middle| \sum_{i=3}^m j_i\gamma_i\right)\right)$$

As the 2nd step of modification, we replace Φ_1 by $\widetilde{\Phi}_1$ and put

$$
{}^{2}\widetilde{F}_{\lambda} \;:=\; e^{\lambda_M}\, q^{\frac{|\lambda|^2}{2(K+h^\vee)}}\; \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\Big(\tau,\;\; -\beta_1,\;\; \beta_1 + \frac{2}{|\gamma_1|^2}\gamma_1\Big)
$$

$$
\times \sum_{j_3,\cdots,j_m\in\mathbf{Z}} \frac{e^{-\sum\limits_{i=3}^{n}(\lambda|\gamma_i)\beta_i + (K+h^\vee)\sum\limits_{i=3}^{m} j_i[\gamma_i + (\gamma_i|\gamma_1)\beta_1]}\, q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=3}^{m} j_i\gamma_i\right|^2 + \sum\limits_{i=3}^{m} j_i(\lambda|\gamma_i)}}{\prod\limits_{p=3}^{n}(1-e^{-\beta_p}\, q^{j_p})}
$$

$$
\times\; \underbrace{\widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_2|^2;(\lambda|\gamma_2)\right]}\Big(\tau,\;\; -\beta_2,\;\; \beta_2 + \frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2 + \frac{2\tau}{|\gamma_2|^2}\big(\gamma_2\big|\sum_{i=3}^{m} j_i\gamma_i\big)\Big)}_{\displaystyle \begin{array}{c} \| \\ e^{(K+h^\vee)\beta_2\left(\gamma_2\middle|\sum\limits_{i=3}^{m} j_i\gamma_i\right)}\,\widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_2|^2;(\lambda|\gamma_2)\right]}\Big(\tau,\;\; -\beta_2,\;\; \beta_2 + \frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2\Big)\end{array}}
$$

$$= e^{\lambda_M} q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;(\lambda|\gamma_1)\right]}\Big(\tau, \ -\beta_1, \ \beta_1 + \frac{2}{|\gamma_1|^2}\gamma_1\Big)$$

$$\times \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_2|^2;(\lambda|\gamma_2)\right]}\Big(\tau, \ -\beta_2, \ \beta_2 + \frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2\Big)$$

$$\times \sum_{j_3,\cdots,j_m \in \mathbf{Z}} \frac{e^{-\sum\limits_{i=3}^{n}(\lambda|\gamma_i)\beta_i + (K+h^\vee)\sum\limits_{i=3}^{n} j_i[\gamma_i + (\gamma_i|\gamma_1)\beta_1 + (\gamma_i|\gamma_2)\beta_2]} \, q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=3}^{m} j_i\gamma_i\right|^2 + \sum\limits_{i=3}^{m} j_i(\lambda|\gamma_i)}}{\prod\limits_{p=3}^{n}(1 - e^{-\beta_p} q^{j_p})}$$

Repeating this procedure n-times, we obtain

$$
\begin{aligned}
{}^{n}\widetilde{F}_{\lambda} &= \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_1|^2;\,(\lambda|\gamma_1)\right]}\left(\tau,\ \ -\beta_1,\ \ \beta_1+\frac{2}{|\gamma_1|^2}\gamma_1\right) \\
&\times\ \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_2|^2;\,(\lambda|\gamma_2)\right]}\left(\tau,\ \ -\beta_2,\ \ \beta_2+\frac{2}{|\gamma_2|^2}\widetilde{\gamma}_2\right) \\
&\qquad\vdots \\
&\times\ \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_n|^2;\,(\lambda|\gamma_n)\right]}\left(\tau,\ \ -\beta_n,\ \ \beta_n+\frac{2}{|\gamma_n|^2}\widetilde{\gamma}_n\right) \\
&\times\ \underbrace{e^{\lambda_M}\, q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{j_{n+1},\cdots,j_m\in\mathbf{Z}} e^{(K+h^\vee)\sum\limits_{i=n+1}^{m} j_i\widetilde{\gamma}_i}\, q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=n+1}^{m} j_i\gamma_i\right|^2+\sum\limits_{i=n+1}^{m} j_i(\lambda|\gamma_i)}}_{\text{classical theta function}}
\end{aligned}
$$

Thus we obtain the modification of F_λ:

$$\widetilde{F}_\lambda = \prod_{p=1}^{n} \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_p|^2;\,(\lambda|\gamma_p)\right]}\Big(\tau,\ -\beta_p,\ \beta_p + \frac{2}{|\gamma_p|^2}\widetilde{\gamma}_p\Big)$$

$$\times\ \underbrace{e^{\lambda_M}\, q^{\frac{|\lambda|^2}{2(K+h^\vee)}} \sum_{j_{n+1},\cdots,j_m\in\mathbf{Z}} e^{(K+h^\vee)\sum\limits_{i=n+1}^{m} j_i\widetilde{\gamma}_i}\ q^{\frac{K+h^\vee}{2}\left|\sum\limits_{i=n+1}^{m} j_i\gamma_i\right|^2 + \sum\limits_{i=n+1}^{m} j_i(\lambda|\gamma_i)}}_{\substack{||\\ (\mathrm{I})}}$$

To compute (I), put $\alpha := \sum\limits_{i=n+1}^{m} j_i\widetilde{\gamma}_i \in M$, then

$$(\mathrm{I}) = \sum_{\alpha\in M} e^{\lambda_M+(K+h^\vee)\alpha}\, q^{\frac{1}{2(K+h^\vee)}|\lambda_M+(K+h^\vee)\alpha|^2} = \Theta_{\lambda_M}$$

hence

$$\widetilde{F}_\lambda = \prod_{p=1}^{n} \widetilde{\Phi}_1^{\left[\frac{K+h^\vee}{2}|\gamma_p|^2;\,(\lambda|\gamma_p)\right]}\Big(\tau,\ -\beta_p,\ \beta_p + \frac{2}{|\gamma_p|^2}\widetilde{\gamma}_p\Big) \times \Theta_{\lambda_M}$$

□

Example: $\widehat{sl}(m|n)\ \ (m>n)$

$$\overline{\Delta}^{+}_{\text{even}} = \{\varepsilon_i - \varepsilon_j,\ \ \delta_i - \delta_j\}_{i<j}$$

$$\overline{\Delta}^{+}_{\text{odd}} = \{\underset{(i\leqq j)}{\varepsilon_i - \delta_j},\ \ \underset{(i<j)}{\delta_i - \varepsilon_j}\}$$

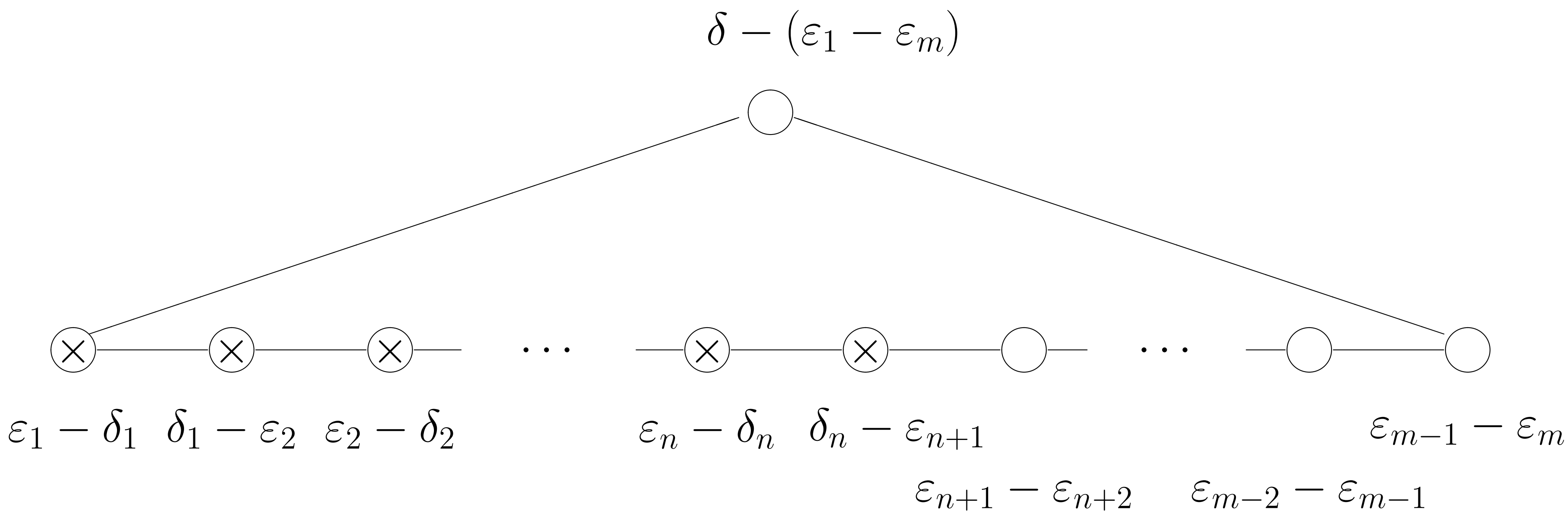
where $\varepsilon_i\ \ (1 \leqq i \leqq m)$ and $\delta_i\ \ (1 \leqq i \leqq n)$

with symmetric inner product $\begin{cases} (\varepsilon_i|\varepsilon_j) = \delta_{i,j} \\ (\delta_i|\delta_j) = -\delta_{i,j} \\ (\varepsilon_i|\delta_j) = 0 \end{cases}$

$$\overline{\Pi} = \{\text{simple roots of } sl(m|n)\}$$

$$= \{\underset{(1\leqq i\leqq n)}{\varepsilon_i - \delta_i},\ \ \underset{(1\leqq i\leqq n)}{\delta_i - \varepsilon_{i+1}},\ \ \underset{(n+1\leqq i\leqq m-1)}{\varepsilon_i - \varepsilon_{i+1}}\}$$

Dynkin diagram of $\widehat{sl}(m|n)$ $(m > n)$



Put

$$\beta_j := \varepsilon_j - \delta_j \qquad (1 \leqq j \leqq n)$$

$$T := \{\beta_1, \ldots, \beta_n\}$$

$$T_{\mathbf{C}} := \bigoplus_{j=1}^{n} \mathbf{C}\beta_j$$

$$M^\sharp := \Big\langle \underset{\substack{\| \; put \\ \gamma_j}}{\varepsilon_m - \varepsilon_j} \; ; \; 1 \leqq j \leqq m \Big\rangle_{\mathbf{Z}} \quad : \quad \text{root lattice of } sl(m, \mathbf{C})$$

$$\overline{W}^! := \left\{ w \in \overline{W}^\sharp \; ; \; w(\beta_j) = \beta_j \;\; ({}^\forall j) \right\}$$

$\{g_i\}_{i \in I}$: a set of representatives of $\overline{W}^\sharp / \overline{W}^!$

i.e, $$\overline{W}^\sharp = \bigcup_{i \in I} g_i \overline{W}^!$$

$$P^K_{+,T} \overset{put}{:=} \left\{ \Lambda \ ; \begin{array}{ll} \text{(a)} & \text{integrable w.r.to } \widehat{sl}(m, \mathbf{C}) \\ \text{(b)} & (\Lambda|\beta) = 0 \quad ({}^\forall \beta \in T) \\ \text{(c)} & (\Lambda|\delta) = K \end{array} \right\}$$

Then

$$P^K_{+,T} = \left\{ \begin{array}{l} K\Lambda_0 + \sum_{i=1}^{n} k_i \beta_i + \sum_{i=n+1}^{m-1} k_i \varepsilon_i \ ; \\ \quad \text{(i)} \quad K, \ k_i \in \mathbf{Z}_{\geqq 0} \\ \quad \text{(ii)} \quad K \geqq k_1 \geqq k_2 \geqq \cdots \geqq k_{m-1} \end{array} \right\}$$

For $\Lambda \in P_{+,T}^{K}$,

$$R^{(-)} \cdot \mathrm{ch}_{\Lambda}^{(-)} = q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}} \sum_{w \in W^{\sharp}} \varepsilon(w) w \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n}(1-e^{-\beta_i})} \right) \qquad (h^{\vee} = m-n)$$

$$= q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}} \sum_{w \in \overline{W}^{\sharp}} \varepsilon(w) w \sum_{\alpha \in M^{\sharp}} t_{\alpha} \left(\frac{e^{\Lambda+\rho}}{\prod_{i=1}^{n}(1-e^{-\beta_i})} \right)$$

$$= \sum_{i \in I} \varepsilon(g_i) g_i \sum_{w \in \overline{W}^{!}} \varepsilon(w) \underbrace{\left(q^{\frac{|\Lambda+\rho|^2}{2(K+h^{\vee})}} \sum_{\alpha \in M^{\sharp}} t_{\alpha} \left(\frac{e^{w(\Lambda+\rho)}}{\prod_{i=1}^{n}(1-e^{-\beta_i})} \right) \right)}_{\substack{\| \\ F_{w(\Lambda+\rho)}}}$$

Define the modified supercharacter by

$$R^{(-)} \cdot \widetilde{\mathrm{ch}}_{\Lambda}^{(-)} := \sum_{i \in I} \varepsilon(g_i) g_i \sum_{w \in \overline{W}^!} \varepsilon(w) \underbrace{\widetilde{F}_{w(\Lambda+\rho)}}_{\substack{\| \\ \Theta_{w(\Lambda+\rho)_M} \times \varphi_T}}$$

$$= \sum_{i \in I} \varepsilon(g_i) g_i \Bigg(\underbrace{\sum_{w \in \overline{W}^!} \varepsilon(w) \Theta_{w(\Lambda_M+\rho_M)}}_{\substack{\| \ put \\ A^!_{\Lambda_M+\rho_M}}} \times \varphi_T \Bigg)$$

where

$$\varphi_T := \prod_{p=1}^{n} \widetilde{\Phi}_1^{[\frac{K+h^\vee}{2}|\gamma_p|^2; (\Lambda+\rho|\gamma_p)]} \Big(\tau, \ \ -\beta_p, \ \ \beta_p + \frac{2}{|\gamma_p|^2} \widetilde{\gamma}_p \Big)$$

$$= \prod_{p=1}^{n} \widetilde{\Phi}_1^{[K+h^\vee]} \Big(\tau, \ \ -\beta_p, \ \ \gamma_p + \sum_{j=1}^{p} \beta_j \Big)$$

and Λ_M and ρ_M are defined by

$$\Lambda = \underbrace{K\Lambda_0 + \sum_{i=n+1}^{m-1} k_i\varepsilon_i}_{\substack{\| \\ \Lambda_M}} + \sum_{i=1}^{n} k_i\beta_i$$

$$\rho = \underbrace{(\overset{\substack{h^\vee \\ \|}}{m - n})\Lambda_0 + \sum_{i=n+1}^{m-1} (m-i)\varepsilon_i}_{\substack{\| \\ \rho_M}} + (m-n-1)\sum_{i=1}^{n} \beta_i$$

<u>Theorem 5.</u> Let $\lambda \in P^K_{+,T}$; then

$$1) \quad R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_\lambda = \sum_{i \in I} \varepsilon(g_i) g_i \left(A^!_{\lambda_M + \rho_M} \prod_{p=1}^{n} \widetilde{\Phi}^{[K+h^\vee]}_1 \Big(\tau, -\beta_p, \gamma_p + \sum_{j=1}^{p} \beta_j \Big) \right)$$

$$2) \quad \big(R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_\lambda\big)\big|_S =$$

$$(-i)^{\frac{m-n-1}{2}} |M^*/(K+h^\vee)M|^{-\frac{1}{2}} \sum_{\substack{\mu \in P^K_{+,T} \\ \mathrm{mod}\,(T_{\mathbf{C}} + \mathbf{C}\delta)}} \left(\sum_{w \in \overline{W}^!} \varepsilon(w) e^{-\frac{2\pi i}{K+h^\vee}(\overline{\lambda+\rho} | w(\overline{\mu+\rho}))} \right) R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_\mu$$

$$3) \quad \big(R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_\lambda\big)\big|_T = e^{\frac{\pi i |\overline{\lambda+\rho}|^2}{K+h^\vee}} R^{(-)} \cdot \widetilde{\mathrm{ch}}^{(-)}_\lambda$$

Remark.

The modified supercharacters $\widetilde{\mathrm{ch}}_\lambda^{(-)}$ are determined

depending on λ mod $T_{\mathbf{C}}$

Namely, for $\lambda, \mu \in P_{+,T}^K$,

$$\lambda \equiv \mu \mod T_{\mathbf{C}} \iff \widetilde{\mathrm{ch}}_\lambda^{(-)} = \widetilde{\mathrm{ch}}_\mu^{(-)}$$

Admissible representations:

$$\Updownarrow \ \text{def}$$

integrable w.r.to a suitable sub-root system

Example. In the case $\widehat{sl}(2|1)$,

$$\Lambda \ : \ \text{admissible} \ \overset{def}{\Longleftrightarrow} \ \begin{array}{l} \Lambda \ : \ \text{integrable w.r.to} \ \{k_i\delta + \alpha_i\}_{i=0,1,2} \\ \qquad\qquad\qquad\qquad \text{or} \quad \{k_i\delta - \alpha_i\}_{i=0,1,2} \end{array}$$

For $\Lambda = K\Lambda_0 - \frac{1}{K}(k_2\alpha_1 + k_1\alpha_2)$ where $K = \frac{m}{M} - 1$ $\left(\begin{array}{rcl} m, \ M & \in & \mathbf{N} \\ (m, M) & = & 1 \end{array}\right)$,

$$R^{(-)}\text{ch}_\Lambda^{(-)} = \Phi^{[m;0]}\Big(M\tau, \ z_1 + k_1\tau, \ z_2 + k_2\tau, \ \frac{t}{M}\Big)$$

Modification:

$$R^{(-)}\widetilde{\text{ch}}_\Lambda^{(-)} = \widetilde{\Phi}^{[m;0]}\Big(M\tau, \ z_1 + k_1\tau, \ z_2 + k_2\tau, \ \frac{t}{M}\Big)$$

Functions $\widetilde{\Psi}^{[M,m;s]}_{i;\,j,k}$ $(i=1,2)$ **:**

$$\widetilde{\Psi}^{[M,m;s]}_{i;\,j,k}(\tau, z_1, z_2, t) \ := \ q^{\frac{mjk}{M}} e^{\frac{2\pi i m}{M}(kz_1+jz_2)} \widetilde{\Phi}^{[m;s]}_{i}\Big(M\tau, z_1+j\tau, z_2+k\tau, \frac{t}{M}\Big)$$

$$\widetilde{\Psi}^{[M,m;s]}_{j,k} \ := \ \widetilde{\Psi}^{[M,m;s]}_{1;\,j,k} \ - \ \widetilde{\Psi}^{[M,m;s]}_{2;\,j,k}$$

Note. $\widetilde{\Psi}^{[M,m;s]}_{i;\,j,k}$ are determined depending on $j, k \bmod M\mathbf{Z}$.

Transformation property :

$$\widetilde{\Psi}^{[M,m;s]}_{i;\,j,k}\Big(-\frac{1}{\tau}, \frac{z_1}{\tau}, \frac{z_2}{\tau}, t-\frac{z_1z_2}{\tau}\Big) = \frac{\tau}{M} \sum_{(a,b)\in\Omega_M} e^{-\frac{2\pi i m}{M}(ak+bj)} \widetilde{\Psi}^{[M,m;s]}_{i;\,a,b}(\tau, z_1, z_2, t)$$

where $\Omega_M \ := \ \big\{(a,b) \in \mathbf{Z}^2 \, ; \, a+b \in 2\mathbf{Z}\big\}/\sim$

$$(a,b) \ \sim \ (a', b') \quad \overset{def}{\Longleftrightarrow} \quad \begin{cases} (a-b)-(a'-b') \ \in \ 2M\mathbf{Z} \\ (a+b)-(a'+b') \ \in \ 2M\mathbf{Z} \end{cases}$$

For complete arguments on modular properties of characters and supercharacters, we need to consider twisted characters and supercharacters.

<u>Twisted characters:</u>

$$\left.\begin{array}{r}\text{non-twisted}\\ \text{twisted}\end{array}\right\}\ \text{affinization of}\ \underset{\mathfrak{g}}{\overset{\text{finite-dimensional Lie superalgebra}}{\downarrow}} \quad :$$

$$\widehat{\mathfrak{g}} \;=\; \Big(\bigoplus_{n\in\mathbf{Z}} t^n\otimes\mathfrak{g}\Big)\oplus\mathbf{C}K\oplus\mathbf{C}d \qquad : \ \text{non-twisted}$$

$$\widehat{\mathfrak{g}}^{tw} \;=\; \Big(\bigoplus_{n\in\mathbf{Z}} t^n\otimes\mathfrak{g}_{\text{even}}\Big)\oplus\Big(\bigoplus_{n\in\frac{1}{2}+\mathbf{Z}} t^n\otimes\mathfrak{g}_{\text{odd}}\Big)\oplus\mathbf{C}K\oplus\mathbf{C}d \ : \quad \text{twisted}$$

Correspondingly, $\left.\begin{array}{r}\text{non-twisted}\\ \text{twisted}\end{array}\right\}$ (super-)characters $\mathrm{ch}^{(\varepsilon)}_{\varepsilon'}$ are defined

where

$$\varepsilon=\begin{cases}0 \ : \ \text{super-character}\\ \frac{1}{2} \ : \ \text{character}\end{cases} \qquad \varepsilon'=\begin{cases}0 \ : \ \text{non-twisted}\\ \frac{1}{2} \ : \ \text{twisted}\end{cases}$$

Functions $\widetilde{\Psi}^{(\pm)[M,m;s,\,\varepsilon]}_{i;\,j,k;\,\varepsilon'}$ $(i=1,2)$ **:**

For $\varepsilon,\, \varepsilon' = 0, \frac{1}{2}$ and $j, k \in \varepsilon' + \mathbf{Z}$, put

$$\widetilde{\Psi}^{(\pm)[M,m;s;\,\varepsilon]}_{i;\,j,k;\,\varepsilon'}(\tau, z_1, z_2, t)$$

$$:= (\pm 1)^{j+\varepsilon'} q^{\frac{mjk}{M}} e^{\frac{2\pi i m}{M}(kz_1+jz_2)} \widetilde{\Phi}^{(\pm)[m;s]}_i \Big(M\tau,\, z_1 + j\tau + \varepsilon,\, z_2 + k\tau + \varepsilon,\, \frac{t}{M} \Big)$$

$$\widetilde{\Psi}^{(\pm)[M,m;s;\,\varepsilon]}_{j,k;\,\varepsilon'} := \widetilde{\Psi}^{(\pm)[M,m;s;\,\varepsilon]}_{1;\,j,k;\,\varepsilon'} - \widetilde{\Psi}^{(\pm)[M,m;s;\,\varepsilon]}_{2;\,j,k;\,\varepsilon'}$$

Modular transformation properties of $\widetilde{\Psi}^{(\pm)[M,m;s,\varepsilon]}_{i;j,k;\varepsilon'}$:

Let $\begin{cases} m & \in & \frac{1}{2}\mathbf{N} \\ M & \in & \mathbf{N}_{\text{odd}} \end{cases}$ s.t. $(M, 2m) = 1$ and $\begin{cases} s & \in & \mathbf{Z} \\ s' & \in & \frac{1}{2}+\mathbf{Z} \end{cases}$. Then

- $$\widetilde{\Psi}^{(\pm)[M,m;s,\varepsilon]}_{i;j,k;\varepsilon'}\Big(-\frac{1}{\tau}, \ \frac{z_1}{\tau}, \ \frac{z_2}{\tau}, \ t-\frac{z_1z_2}{\tau}\Big)$$
$$= \frac{\tau}{M}\sum_{(a,b)\in\Omega_M} e^{-\frac{2\pi im}{M}[(a+\varepsilon)k+(b-\varepsilon)j]} \times \begin{cases} \widetilde{\Psi}^{(+)[M,m;s,\varepsilon']}_{i;a+\varepsilon,b-\varepsilon;\varepsilon}(\tau,z_1,z_2,t) & \text{if} \quad \text{“}+\text{”} \\ \widetilde{\Psi}^{(+)[M,m;s',\varepsilon']}_{i;a+\varepsilon,b-\varepsilon;\varepsilon}(\tau,z_1,z_2,t) & \text{if} \quad \text{“}-\text{”} \end{cases}$$

- $$\widetilde{\Psi}^{(\pm)[M,m;s',\varepsilon]}_{i;j,k;\varepsilon'}\Big(-\frac{1}{\tau}, \ \frac{z_1}{\tau}, \ \frac{z_2}{\tau}, \ t-\frac{z_1z_2}{\tau}\Big)$$
$$= \frac{\tau}{M}\sum_{(a,b)\in\Omega_M} e^{-\frac{2\pi im}{M}[(a+\varepsilon)k+(b-\varepsilon)j]} \times \begin{cases} \widetilde{\Psi}^{(-)[M,m;s,\varepsilon']}_{i;a+\varepsilon,b-\varepsilon;\varepsilon}(\tau,z_1,z_2,t) & \text{if} \quad \text{“}+\text{”} \\ \widetilde{\Psi}^{(-)[M,m;s',\varepsilon']}_{i;a+\varepsilon,b-\varepsilon;\varepsilon}(\tau,z_1,z_2,t) & \text{if} \quad \text{“}-\text{”} \end{cases}$$

In the case $\widehat{sl}(2|1)$, admissible weights of level $K = \frac{m}{M} - 1$ are parametrized by

$$\Omega_{M,0} \ := \ \left\{ (j,k) \in \mathbf{Z}^2 \ ; \ 0 \leqq j,k < M \right\}$$

To describe the <u>twisted characters</u>, define the set

$$\Omega_{M,\frac{1}{2}} \ := \ \left\{ \left(j + \frac{1}{2}, \, k + \frac{1}{2} \right) \ ; \ (j,k) \in \Omega_{M,0} \right\}$$

Modified admissible characters :

For $\varepsilon, \varepsilon' = 0, \frac{1}{2}$ and $(j,k) \in \Omega_{M,\varepsilon'}$,

$$\widetilde{\mathrm{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) \ := \ \frac{\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau,\ z_1,\ z_2,\ t)}{R^{(\varepsilon)}_{\varepsilon'}(\tau,\ z_1,\ z_2,\ t)}$$

where

$$\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z_1, z_2, t) \ := \ q^{\frac{mjk}{M}} \, e^{\frac{2\pi im}{M}(kz_1+jz_2)} \widetilde{\Phi}^{[m]}\Big(M\tau, z_1 + j\tau + \varepsilon, z_2 + k\tau + \varepsilon, \frac{t}{M}\Big)$$

$$R^{(\varepsilon)}_{\varepsilon'}(\tau,\ z_1,\ z_2,\ t) \ := \ (-1)^{2\varepsilon(1-2\varepsilon')} \, e^{2\pi it} \, \frac{\eta(\tau)^3 \, \vartheta_{11}(\tau, z_1 + z_2)}{\vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_1) \, \vartheta_{1-2\varepsilon', 1-2\varepsilon}(\tau, z_2)}$$

and $\vartheta_{00}, \vartheta_{01}, \vartheta_{10}, \vartheta_{11}$ are theta functions in the Mumford's book "Tata lectures on theta I".

Modular transformation of $\widehat{sl}(2|1)$-admissible characters :

Let $\begin{cases} m \in \mathbf{N}_{\geqq 2} \\ \gcd(M,\, 2m) = 1 \end{cases}$ or $\begin{cases} m = 1 \\ M \in \mathbf{N} \end{cases}$. Then

$$\widetilde{\text{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}|_S \;=\; \frac{(-1)^{4\varepsilon\varepsilon'}}{M} \sum_{(a,b)\,\in\,\Omega_{M,\varepsilon}} e^{-\frac{2\pi i m}{M}(ka+jb)}\, \widetilde{\text{ch}}^{[M,m;\varepsilon']}_{a,b;\varepsilon}$$

$$\widetilde{\text{ch}}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}|_T \;=\; e^{\frac{2\pi i m}{M}jk - \pi i\varepsilon'}\, \widetilde{\text{ch}}^{[M,m;\varepsilon+\varepsilon']}_{j,k;\varepsilon'}$$

where $(j,k) \in \Omega_{M,\varepsilon'}$.

Quantum Hamiltonian reduction :

$$\begin{cases} \mathfrak{g} & : \text{ finite-dim Lie superalgebra} \\ f & : \text{ nilpotent element} \\ K & : \text{ level} \end{cases}$$

$$\Downarrow$$

$$W(\mathfrak{g}, f, K) \quad : \quad \text{W-algebra}$$

Example:

$$\left.\begin{array}{rcl} \mathfrak{g} & = & sl(2|1) \\ f & = & e_{-\theta} \end{array}\right\} \implies \text{N=2 SCA}$$

$$(\theta \;=\; \text{highest root})$$

<u>N=2 SCA</u> is spanned by

$$\begin{cases} L_n, \quad J_n \quad (n \in \mathbf{Z}) & : \quad \text{even elements} \\ G_n^{\pm} \quad (n \in \varepsilon + \mathbf{Z}) & : \quad \text{odd elements} \\ c & : \quad \text{central element} \end{cases} \qquad \varepsilon = \begin{cases} \frac{1}{2} & : \text{Neveu-Schwarz} \\ 0 & : \text{Ramond} \end{cases}$$

with (anti-)commutation relations

	L_n	J_n	G_n^+	G_n^-
L_m	$(m-n)L_{m+n}$ $+\frac{m^3-m}{12}\delta_{m+n,0}c$	$-nJ_{m+n}$	$(\frac{m}{2}-n)G_{m+n}^+$	$(\frac{m}{2}-n)G_{m+n}^-$
J_m	mJ_{m+n}	$\frac{m}{3}\delta_{m+n,0}c$	G_{m+n}^+	$-G_{m+n}^-$
G_m^+	$(m-\frac{n}{2})G_{m+n}^+$	$-G_{m+n}^+$	0	$L_{m+n}+\frac{m-n}{2}J_{m+n}$ $+\frac{c}{6}(m^2-\frac{1}{4})\delta_{m+n,0}$
G_m^-	$(m-\frac{n}{2})G_{m+n}^-$	G_{m+n}^-	$L_{m+n}-\frac{m-n}{2}J_{m+n}$ $+\frac{c}{6}(m^2-\frac{1}{4})\delta_{m+n,0}$	0

Note : Cartan subalgebra $=$ linear span of L_0, J_0, c

Characters of N=2 h.w. reps via quantum reduction:

The characters of N=2 SCA are obtained by letting $\begin{cases} z_1 = z \\ z_2 = -z \end{cases}$

in the characters of $\widehat{sl}(2|1)$-modules:

$$\widetilde{\chi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z, t) \;:=\; e^{2\pi i t c_{M,m}} \, \frac{\widetilde{\Psi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau, z, -z, 0)}{R^{\mathrm{N}=2(\varepsilon)}_{\varepsilon'}(\tau, z, 0)}$$

where

$$(j,k) \;\in\; \Omega^{[N=2]}_{M;\varepsilon'} := \left\{ (j,k) \;\middle|\; \begin{array}{l} j,k \;\in\; \varepsilon' + \mathbf{Z}_{\geqq 0} \\ 0 \;<\; j,\, j+k \;\leqq\; M-1 \end{array} \right\}$$

$$c_{M,m} \;:=\; 3\Big(1 - \frac{2m}{M}\Big) \quad : \quad \text{central charge}$$

<u>Theorem 6.</u> (modular properties of N=2 characters)

$$\widetilde{\chi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}\Big(-\frac{1}{\tau},\frac{z}{\tau},t-\frac{z^2}{6\tau}\Big) = \sum_{(a,b)\in\Omega^{[N=2]}_{M;\varepsilon}} S^{[M,m]}_{(j,k),(a,b)}\ \widetilde{\chi}^{[M,m;\varepsilon']}_{a,b;\varepsilon}(\tau,z,t)$$

$$\widetilde{\chi}^{[M,m;\varepsilon]}_{j,k;\varepsilon'}(\tau+1,z,t) \ = \ e^{\frac{2\pi i m}{M}jk-\frac{\pi i\varepsilon'}{2}}\ \widetilde{\chi}^{[M,m;\,\varepsilon+\varepsilon']}_{j,k;\varepsilon'}(\tau,z,t)$$

where

$$S^{[M,m]}_{(j,k),(a,b)} := (-i)^{(1-2\varepsilon)(1-2\varepsilon')}\,\frac{2}{M}\,e^{\frac{2\pi i m}{M}(j-k)(a-b)}\ \sin\frac{m}{M}(j+k)(a+b)\pi$$

Fusion coefficients of N=2 SCA:

$$\mathfrak{F}_M^{[N=2]} \quad := \bigcup_{\varepsilon+\varepsilon'+\varepsilon''= \ \frac{1}{2} \ \text{or} \ \frac{3}{2}} \left(\Omega^{[N=2]}_{M,\varepsilon} \times \Omega^{[N=2]}_{M,\varepsilon'} \times \Omega^{[N=2]}_{M,\varepsilon''} \right)$$

For $(\lambda, \mu, \nu) \in \mathfrak{F}_M^{[N=2]}$,

$$N^{N=2,[M,m]}_{\lambda,\mu,\nu} \quad \overset{put}{:=} \sum_{\xi \,\in\, \Omega^{[N=2]}_{M,\frac{1}{2}}} \frac{S^{[M,m]}_{\lambda,\xi} \ S^{[M,m]}_{\mu,\xi} \ S^{[M,m]}_{\nu,\xi}}{S^{[M,m]}_{(\frac{1}{2},\frac{1}{2}),\xi}}$$

Theorem 7. Let $\lambda := (j,k),\ \mu := (j',k'),\ \nu := (j'',k''),$ then

1) $N_{\lambda,\mu,\nu}^{N=2[M,m]} = 0$ or $1.$

2) $N_{\lambda,\mu,\nu}^{N=2[M,m]} = 1 \iff$ (F1) or (F2)

$$
\text{(F1)} \quad \begin{cases} (j+j'+j'') - (k+k'+k'') = 0 \\ |(j'+k') - (j''+k'')| < j+k < (j'+k') + (j''+k'') \\ (j+j'+j'') + (k+k'+k'') < 2M \end{cases}
$$

$$
\text{(F2)} \quad \begin{cases} (j+j'+j'') - (k+k'+k'') = \pm M \\ |(j'+k') - (j''+k'')| < M-j-k < (M-j'-k') + (M-j''-k'') \\ (j+j'+j'') + (k+k'+k'') > M \end{cases}
$$

Let

$$h \ := \ \text{eigenvalue of } \ L_0 \ \text{ on the h.w.vector}$$

$$s \ := \ \text{eigenvalue of } \ J_0 \ \text{ on the h.w.vector}$$

- N=2 highest weight representations are characterized by $(h,\ s,\ c)$

- $$h_{j,k;\varepsilon} \ = \ \frac{m}{M}\Big(jk - \frac{1}{4}\Big) - \frac{1+2\varepsilon}{8}$$

- $$s_{j,k;\varepsilon} \ = \ \frac{m}{M}(k-j) - \frac{1-2\varepsilon}{2}$$

- $$c_{M,m} \ = \ \frac{3(M-2m)}{M} \ = \ 3 \cdot \frac{\frac{M}{m}-2}{\left(\frac{M}{m}-2\right)+2}$$

Note:

- $m = 1 \quad \Longrightarrow \quad c_{M,1} \;=\; 3 \cdot \dfrac{M-2}{(M-2)+2} \qquad (M \in \mathbf{N}_{\geqq 2})$

  (well known **minimal series** = unitary series)

- $\left.\begin{array}{c} m \geqq 2 \\ (M,\, 2m) = 1 \end{array}\right\} \Longrightarrow$ **mock modular series**

Similar method for quantum reduction works

for all affine Lie superalgebras

to give

new series of mock modular representations

for N=3, N=4, ... superconformal algebras.

Natural Problem (Open):

What is the representation theoretical meaning

of the additional terms ?

and

of the Zwegers type functions ?