Moonshine: Lecture 3

Ken Ono (Emory University)

I'm going to talk about...

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I. History of Moonshine

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II. Distribution of Monstrous Moonshine

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II. Distribution of Monstrous Moonshine

III. Umbral Moonshine

The Monster

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Conjecture (Fischer and Griess (1973))

There is a huge simple group (containing a double cover of Fischer's B) with order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

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Theorem (Griess (1982))

The Monster group $\mathbb{M}$ *exists.*

Classification of Finite Simple Groups

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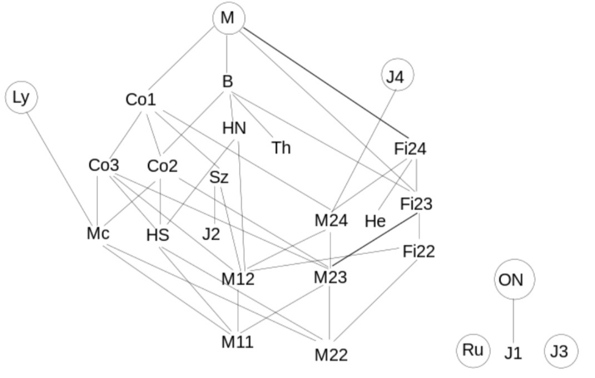

Ogg's Theorem

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Corollary (Ogg, 1974)

Toutes les valuers supersingulières de j sont $\mathbb{F}_p$ si, et seulement si, $g^+ = 0$,

i.e. $p \in Ogg_{ss} := \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}$.

Ogg's Jack Daniels Problem

Remarque 1. – Dans sa leçon d'ouverture au Collège de France, le 14 janvier 1975, J. TITS mentionna le groupe de Fischer, "le monstre", qui, s'il existe, est un groupe simple "sporadique" d'ordre

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71 \,,$$

i. e. divisible exactement par les quinze nombres premiers de la liste du corollaire. Une bouteille de Jack Daniels est offerte à celui qui expliquera cette coïncidence.

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Remark

This is the first hint of **Moonshine**.

Second hint of moonshine

John McKay observed that

$$196884 = 1 + 196883$$

John Thompson's generalizations

Thompson further observed:

$$\underbrace{\begin{array}{r} 196884 \\ 21493760 \\ 864299970 \end{array}}_{\text{Coefficients of } j(\tau)} \begin{array}{c} = \\ = \\ = \end{array} \underbrace{\begin{array}{l} 1 + 196883 \\ 1 + 196883 + 21296876 \\ 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{array}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

Klein's j-function

Definition

Klein's j-function

$$\begin{aligned} j(\tau) - 744 &= \sum_{n=-1}^{\infty} c(n) q^n \\ &= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots \end{aligned}$$

satisfies

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau) \quad \text{for every matrix } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

$$\chi_1(e) = 1$$
$$\chi_2(e) = 196883$$
$$\chi_3(e) = 21296876$$
$$\chi_4(e) = 842609326$$
$$\vdots$$
$$\chi_{194}(e) = 258823477531055064045234375.$$

Monster module

Conjecture (Thompson)

There is an infinite-dimensional graded module

$$V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$$

with

$$\dim(V_n^\natural) = c(n).$$

The McKay-Thompson Series

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{tr}(g|V_n^\natural)q^n.$$

Conway and Norton

Conjecture (Monstrous Moonshine)

For each $g \in \mathbb{M}$ there is an explicit genus 0 discrete subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$

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Conjecture (Monstrous Moonshine)

For each $g \in \mathbb{M}$ there is an explicit genus 0 discrete subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the unique modular function with

$$T_g(\tau) = q^{-1} + O(q).$$

Borcherds' work

Theorem (Frenkel–Lepowsky–Meurman)

The moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a vertex operator algebra whose graded dimension is given by* $j(\tau) - 744$*, and whose automorphism group is* $\mathbb{M}$.

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The Monstrous Moonshine Conjecture is true.

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The Monstrous Moonshine Conjecture is true.

Remark

Earlier work of Atkin, Fong and Smith numerically confirmed Monstrous moonshine.

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Question A

Do order p elements in $\mathbb{M}$ know the $\overline{\mathbb{F}}_p$ supersingular j-invariants?

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Theorem (Dwork's Generating Function)

If $p \geq 5$ is prime, then

$$(j(\tau) - 744) \mid U(p) \equiv$$
$$-\sum_{\alpha \in SS_p} \frac{A_p(\alpha)}{j(\tau) - \alpha} - \sum_{g(x) \in SS_p^*} \frac{B_p(g)j(\tau) + C_p(g)}{g(j(\tau))} \pmod{p}.$$

Answer to Question A

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- ....giving us Dwork's generating function

$$T_g \mid U(p) \equiv (j - 744) \mid U(p) \pmod{p}. \qquad \square$$

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Question B

If $p \notin Ogg_{ss}$, then why do we expect $p \nmid \#\mathbb{M}$?

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Answer

- *By Moonshine, if* $g \in \mathbb{M}$ *has order* p*, then* $\Gamma_g \subset \Gamma_0^+(p)$ *has genus 0.*

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If $p \notin Ogg_{ss}$, then why do we expect $p \nmid \#\mathbb{M}$?

Answer

- *By Moonshine, if $g \in \mathbb{M}$ has order p, then $\Gamma_g \subset \Gamma_0^+(p)$ has genus 0.*
- *By Ogg, if $p \notin Ogg_{ss}$, then $X_0^+(p)$ has positive genus.*

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Heuristic Argument?

- *Let $h_p(\tau)$ be the hauptmodul for $\Gamma_0^+(p)$.*
- *Hecke implies that $h_p \mid U(p) \equiv (j - 744) \mid U(p) \pmod{p}$.*
- *Deligne for E_{p-1} gives $h_p \mid U(p) \in S_{p-1}(1) \pmod{p}$.*
- *Implies $j'(h_p \mid U(p)) \in S_{p+1}(1) \pmod{p}$.*
- *Moonshine implies $j'(h_p \mid U(p))$ comes from Θ's.*
- *But Serre implies $j'(h_p \mid U(p)) \in S_2(p) \pmod{p}$.*
- *We expect $S_2(p) \pmod{p}$ to be spanned by Θ's.*
- *Pizer proved Θ's from quaternion alg's suffice iff $p \in Ogg_{ss}$.*

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Witten's Conjecture (2007)

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Question (Witten)

How are these different kinds of black hole states distributed?

Distribution of Monstrous Moonshine

Open Problem

Question

Consider the moonshine expressions

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\ &\vdots \\ c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e) \end{aligned}$$

How many '1's, '196883's, etc. show up in these equations?

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
-1	1	0	$\cdots$	0
1	1/2	1/2	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$

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100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
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180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
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Distribution of Moonshine

Theorem 1 (Duncan, Griffin, O)

We have Rademacher style exact formulas for $\mathbf{m}_i(n)$ *of the form*

$$\mathbf{m}_i(n) = \sum_{\chi_i} \sum_{c} \text{Kloosterman sums} \times I\text{-Bessel fcns}$$

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Remark

The dominant term gives

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

Distribution

Remark

We have that

$$\delta\left(\mathbf{m}_i\right) := \lim_{n \to +\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)}$$

is well defined

Distribution

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is well defined, and

$$\delta\left(\mathbf{m}_i\right) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

Orthogonality

Fact

If G is a group and $g, h \in G$, then

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)| & \text{If } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise,} \end{cases}$$

where $C_G(g)$ is the centralizer of g in G.

$T_\chi(\tau)$

- Define

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- From this we can work out that

$$T_{\chi_i}(\tau) = \sum_{n=-1}^{\infty} \mathbf{m}_i(n) q^n.$$

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3. *We can then build a form with matching poles via Poincaré series.*
4. *The Poincaré series give exact formulas for coefficients.*

Umbral (shadow) Moonshine

Present day moonshine

Observation (Eguchi, Ooguri, Tachikawa (2010))

There is a **mock modular form**

$$H(\tau) = q^{-\frac{1}{8}}\left(-2 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \ldots\right).$$

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The degrees of the irreducible repn's of the Mathieu group M_{24} are:

$$1, 23, 45, 231, 252, 253, 483, 770, 990, 1035,$$
$$1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

Mathieu Moonshine

Theorem (Gannon (2013))

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- *Computed using wgt 1/2 weakly holomorphic modular forms.*
- *Integrality follows from "theory of modular forms mod p".*
- *Non-negativity follows from "effectivizing" argument of Bringmann-O on Ramanujan's $f(q)$ mock theta function.*

What are mock modular forms?

Notation. Throughout, let

$$\tau = x + iy \in \mathbb{H} \quad \text{with} \quad x, y \in \mathbb{R}.$$

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Harmonic Maass forms

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A *harmonic Maass form of weight* k *on a subgroup* $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $M : \mathbb{H} \to \mathbb{C}$ satisfying:

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2. We have that $\Delta_k M = 0$.

Fourier expansions

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Fundamental Lemma

If $M \in H_{2-k}$ *and* $\Gamma(a,x)$ *is the incomplete* Γ*-function, then*

$$M(\tau) = \sum_{n \gg -\infty} c^{+}(n) q^{n} + \sum_{n<0} c^{-}(n) \Gamma(k-1, 4\pi|n|y) q^{n}.$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Holomorphic part M^{+} **Nonholomorphic part** M^{-}

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- *We call* M^{+} *a* **mock modular form**.
- *If* $\xi_{2-k} := 2iy^{2-k} \overline{\frac{\partial}{\partial \overline{\tau}}}$, *then the* **shadow of** M *is* $\xi_{2-k}(M^{-})$.

Shadows are modular forms

Fundamental Lemma

The operator $\xi_{2-k} := 2iy^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}}$ *defines a* **surjective** *map*

$$\xi_{2-k} : H_{2-k} \longrightarrow S_k.$$

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Remark

In M_{24} *Moonshine, the McKay-Thompson series are mock modular forms with* **classical Jacobi theta series** *shadows!*

Larger Framework of Moonshine?

Remark

There are well known connections with even unimodular positive definite rank 24 lattices:

$$\mathbb{M} \quad \longleftrightarrow \quad \textbf{Leech lattice}$$

$$M_{24} \quad \longleftrightarrow \quad A_1^{24} \text{ lattice.}$$

Umbral Moonshine Conjecture

Conjecture (Cheng, Duncan, Harvey (2013))

Let L^X (up to isomorphism) be an even unimodular positive-definite rank 24 lattice, and let :

- *X be the corresponding ADE-type root system.*

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Then there is an infinite dimensional graded G^X module K^X for which $H_g^X(\tau)$ is the McKay-Thompson series for g.

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- For $X = A_2^{12}$ we have $G^X = M_{24}$ and Gannon's Theorem.
- There are 22 other isomorphism classes of X, the $H_g^X(\tau)$ constructed from X and its Coxeter number $m(X)$.

Remark

Apart from the Leech case, the $H_g^X(\tau)$ are always weight 1/2 mock modular forms whose shadows are weight 3/2 cuspidal theta series with level $m(X)$.

Our results....

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Theorem 3 (Duncan, Griffin, Ono)

The Umbral Moonshine Conjecture is true.

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The Umbral Moonshine Conjecture is true.

Remark

This result is a "numerical proof" of Umbral moonshine. It is analogous to the work of Atkin, Fong and Smith in the case of monstrous moonshine.

Beautiful examples

Beautiful examples

Example

For M_{12} the MT series include Ramanujan's mock thetas:

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$\phi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$\chi(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})}$$

Strategy of Proof

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For each X we compute non-negative integers $\mathbf{m}_i^X(n)$ for which

$$K^X = \sum_{n=-1}^{\infty} \sum_{\chi_i} \mathbf{m}_i^X(n) V_{\chi_i}.$$

Orthogonality

Fact

If G is a group and $g, h \in G$, then

$$\sum_{\chi_i} \chi_i(g)\overline{\chi_i(h)} = \begin{cases} |C_G(g)| & \text{If } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise,} \end{cases}$$

where $C_G(g)$ is the centralizer of g in G.

$T^X_\chi(\tau)$

- Define the weight 1/2 harmonic Maass form

$$T^X_{\chi_i}(\tau) := \frac{1}{|\mathbb{M}|} \sum_{g \in G^X} \overline{\chi_i(g)} H^X_g(\tau).$$

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We only need to establish **integrality** and **non-negativity**!

Difficulties

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- We have that

$$T^X_{\chi_i}(\tau) = \text{“period integral of a } \Theta\text{-function”} + \sum_{n=-1}^{\infty} \mathbf{m}_i^X(n) q^n.$$

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- Method of **holomorphic projection** gives:

$$\pi_{hol} : H_{\frac{1}{2}} \longrightarrow \widetilde{M}_2 = \{\text{wgt 2 quasimodular forms}\}.$$

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2. $a_f(n, y) = O(y^{2-k})$ as $y \to 0$ for all $n > 0$.

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where for $n > 0$ we have

$$c(n) = \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_f(n, y) e^{-4\pi n y} y^{k-2} dy.$$

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Remark

Holomorphic projections appeared earlier in works of Sturm, and Gross-Zagier, and work of Imamoglu, Raum, and Richter, Mertens, and Zwegers in connection with mock modular forms.

Sketch of the proof of umbral moonshine

- Compute each wgt 1/2 harmonic Maass form $T^{X}_{\chi_i}(\tau)$.

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- Check the finitely many (less than 400) cases directly.

  $\square$

Distribution of Monstrous Moonshine

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Remark

We have that

$$\delta(\mathbf{m}_i) = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)} = \frac{\dim(\chi_i)}{5844076785304502808013602136}.$$

Umbral Moonshine

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Do the infinite dimensional graded G^X modules K^X exhibit deep structure?
Probably....and some work of Duncan and Harvey makes use of indefinite theta series to obtain a VOA structure in the M_{24} case.